

# Math 377 (Real Analysis) Lecture Notes

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Notes for the course MATH 377, instructed by Professor Joseph Benson.

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# 1 September 4, 2019

## 1.1 Basic Notations

A *set* is any collection of objects. These objects are referred to as the *elements* of the set. Given a set  $A$ , we write  $x \in A$  if  $x$  is an element of  $A$ . If  $x$  is not an element of  $A$ , then we write  $x \notin A$ . Given two sets  $A$  and  $B$ , the *union* is written  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$  and the *intersection* is written  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ .

Common sets are:

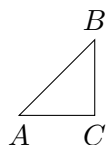
- $\mathbb{N} = \{1, 2, 3, \dots\}$ : the set of natural numbers.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ : the set of integers.
- $\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$ : the set of rationals.
- $\emptyset$ : the empty set;  $\emptyset = \{\}$ .

The inclusion relationship  $A \subseteq B$  is used to indicate that every element of  $A$  is also an element of  $B$ . To assert that  $A = B$  means that  $A \subseteq B$  and  $B \subseteq A$ .

## 1.2 Discussion: The Irrationality of $\sqrt{2}$

Fun facts:

- Greeks: “all numbers are rational.”
- The Pythagorean Theorem says: “a right triangle can have sides  $1, 1, \sqrt{2}$ .”



### Theorem: 1.2.1

$\sqrt{2}$  is not a rational number.

*Proof.* Proof by contradiction. (Idea: Assume the hypothesis to be false, proceed logically until you find a logical discrepancy.)

Assume, for contradiction, that  $\sqrt{2}$  is rational. Then we can write  $\sqrt{2} = \frac{p}{q}$ , where  $p, q \in \mathbb{Z}$  have no common factors. Taking the square of the above equation, we get

$$2 = \frac{p^2}{q^2}. \quad (1)$$

The equation (1) implies

$$2q^2 = p^2. \quad (2)$$

From this, we see that the integer  $p^2$  is even, and hence  $p$  must be even as well. Then we can write  $p = 2r$ , where  $r$  is an integer. Substituting  $2r$  for  $p$  in equation (2), we have

$$2q^2 = (2r)^2 \implies q^2 = 2r^2.$$

Thus  $q^2$  is even, and hence  $q$  is even. At the start, we assumed  $p$  and  $q$  have no common divisor; however, the above logic leads to a contradiction. Therefore,  $\sqrt{2}$  is not rational.  $\square$

### 1.3 The Real Numbers

**Definition 1.**  $\mathbb{R}$ —*real numbers*: “the rationals plus the gaps between the rationals.”

### 1.4 Functions

**Definition 2.** A function  $f$  from  $A$  to  $B$  (write:  $f: A \rightarrow B$ ) is a rule that assigns to each element of  $A$  a single element of  $B$ .

The set  $A$  is called the domain of  $f$ . The range of  $f$  is  $\{f(x) \mid x \in A\} \subseteq B$ .

#### Example: 1.4.1

**Dirichlet’s Function:**

$$g(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q}, \\ 1, & \text{if } x \in \mathbb{Q}. \end{cases}$$

The range of  $g$  is  $\{0, 1\}$ , and the domain of  $g$  is all of  $\mathbb{R}$ .

#### Example: 1.4.2

**Absolute Value Function:**

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

With respect to *addition*, the absolute value satisfies the triangle inequality: if  $a, b \in \mathbb{R}$ , then  $|a + b| \leq |a| + |b|$ .

## 2 September 6, 2019

Today's big question: How to define/construct  $\mathbb{R}$ ?

### 2.1 The Axiom of Completeness

#### Theorem: 2.1.1

**Axiom of Completeness (AoC).** Every nonempty set that is bounded above has a least upper bound.

**Definition 3.** A set  $A$  is bounded above if there  $\exists b \in \mathbb{R}$  s.t.  $a \leq b \forall a \in A$ . We call  $b$  an upper bound of the set  $A$ .

**Definition 4.** A set  $A$  is bounded below: similarly,  $l$  is a lower bound if  $l \leq a \forall a \in A$ .

**Definition 5.** A real number  $s$  is the least upper bound (supremum) of a set  $A$  if:

1.  $s$  is an upper bound of  $A$ ;
2. if  $b$  is any other upper bound of  $A$ , then  $s \leq b$ .

We will always write  $s = \sup A$  for the least upper bound.

**Definition 6.** The greatest lower bound (infimum) is defined analogously. Similarly, the notation is  $l = \inf A$ .

#### Example: 2.2.1

Let

$$A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}.$$

The set  $A$  is bounded above and below. By the definition of the supremum and infimum,  $\sup A = 1$  and  $\inf A = 0$ .

**Definition 7.** A real number  $a_0 \in A$  is a maximum of  $A$  if  $a_0 \geq a$  for all  $a \in A$ . Similarly, a number  $a_1$  is a minimum of  $A$  if  $a_1 \in A$  and  $a_1 \leq a$  for every  $a \in A$ .

Note: when a maximum exists, it is the *supremum*.

**Remark 8.** AoC is not true when considering  $\mathbb{Q}$ .

**Example: 2.2.2**

To illustrate the remark, consider

$$S = \{x \in \mathbb{Q} \mid x^2 < 2\}.$$

The definition of the supremum has two parts. Part (i) says that  $\sup A$  must be an upper bound, and part (ii) states that it must be the smallest one. The following lemma offers an alternative way to restate part (ii).

**Lemma 9.** *Assume  $s \in \mathbb{R}$  is an upper bound for a set  $A$ . Then  $s = \sup A$  if and only if (iff) for any  $\epsilon > 0$ , there exists  $a \in A$  s.t.  $s - \epsilon < a$ .*

*Proof.* ( $\implies$ )

Assume  $s = \sup A$  and let  $\epsilon > 0$  be given arbitrarily. Then  $s - \epsilon < s$ . By part (ii) of the definition,  $s - \epsilon$  is not an upper bound of  $A$ . Therefore, there exists  $a \in A$  with  $s - \epsilon < a$ .

( $\impliedby$ )

Conversely, assume  $s$  is an upper bound s.t. for any  $\epsilon > 0$ ,  $s - \epsilon$  is no longer an upper bound for  $A$ . This implies that every  $b < s$  is not an upper bound. To show  $s = \sup A$ , we need to verify (ii); but this is exactly what we just did. Hence, if  $b$  is any other upper bound for  $A$ , then  $s \leq b$ .  $\square$

### 3 September 9, 2019

First, recall the AoC: every nonempty set that is bounded above has a least upper bound.

#### 3.1 Consequences of Completeness

##### Theorem: 3.1.1

**Nested Interval Property (NIP).** For each  $n \in \mathbb{N}$ , assume we have an interval  $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$  and that  $I_{n+1}$  is a subset of  $I_n$ . Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is,  $\bigcap_{n=1}^{\infty} I_n$  is nonempty.

*Proof.* (Constructive proof)

Since AoC is a statement about bounded sets, consider the set  $A = \{a_n \mid n \in \mathbb{N}\}$  of left-hand endpoints of the intervals. Because the intervals are nested,  $A$  is bounded above by each  $b_n$ ; in particular, each  $b_n$  is an upper bound of  $A$ .

By AoC, there exists  $\sup A$ , which we call  $x = \sup A$ . Now, fix an arbitrary  $n$  and consider the interval  $I_n = [a_n, b_n]$ . Because  $x$  is an upper bound for  $A$ , we have  $a_n \leq x$ ; and since  $b_n$  is an upper bound of  $A$ , we also have  $x \leq b_n$ . Altogether,  $a_n \leq x \leq b_n$ .

Since  $n$  was arbitrary,  $a_n \leq x \leq b_n$  for all  $n \in \mathbb{N}$ . Hence,  $x \in \bigcap_{n=1}^{\infty} I_n$ , and the intersection is nonempty.  $\square$

#### 3.2 The Density of $\mathbb{Q}$ in $\mathbb{R}$

##### Theorem: 3.2.1

**Archimedean Property (AP).**

1. Given any number  $x \in \mathbb{R}$ , there exists an  $n \in \mathbb{N}$  satisfying  $x < n$ .
2. Given any real number  $y > 0$ , there exists  $n \in \mathbb{N}$  such that  $\frac{1}{n} < y$ .

*Proof.* (Proof by contradiction)

Part (i) of the theorem states that  $\mathbb{N}$  is not bounded above. For contradiction, assume that  $\mathbb{N}$  is bounded above. AoC implies  $\exists \sup \mathbb{N} = \alpha$ . Since  $\alpha - 1$  is not an upper bound, there exists  $n \in \mathbb{N}$  s.t.  $\alpha - 1 < n$ . Then  $\alpha < n + 1 \in \mathbb{N}$ , contradicting the fact that  $\alpha$  is an upper bound for  $\mathbb{N}$ .

For part (ii), let  $y > 0$  be given and set  $x = \frac{1}{y}$ . By part (i), there exists  $n \in \mathbb{N}$  such that  $x < n$ . Equivalently,  $\frac{1}{y} < n \implies 0 < \frac{1}{n} < y$ .  $\square$

**Theorem: 3.2.2**

**Density of  $\mathbb{Q}$  in  $\mathbb{R}$ .** For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists  $r \in \mathbb{Q}$  s.t.  $a < r < b$ . That is, the rationals are dense in  $\mathbb{R}$ .

*Proof.* Given  $a < b$ , we want to find integers  $m$  and  $n$  such that  $a < \frac{m}{n} < b$ , where  $n \in \mathbb{N}$ . By AP (ii), choose  $n > 0$  so that  $\frac{1}{n} < b - a$ . Let  $m = \lfloor na \rfloor + 1$ . Then  $na < m \leq na + 1 < nb$  (since  $n(b - a) > 1$ ), hence

$$a < \frac{m}{n} < b.$$

$\square$

### 3.3 The Existence of Square Roots

**Theorem: 3.3.1**

There exists a real number  $x \in \mathbb{R}$  such that  $x^2 = 2$ .

*Proof.* Consider the set  $T = \{t \in \mathbb{R} \mid t^2 < 2\}$  and set  $\alpha = \sup T$ . We will show that both possibilities  $\alpha^2 < 2$  and  $\alpha^2 > 2$  lead to a contradiction.

First, assume  $\alpha^2 < 2$ . Our goal is to show that this violates  $\alpha$  being an upper bound. Note that, for  $n \geq 1$ ,

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n}.$$

By AP (ii), choose  $n$  such that  $\frac{1}{n} < \frac{2-\alpha^2}{2\alpha+1}$ . Then

$$\left(\alpha + \frac{1}{n}\right)^2 < \alpha^2 + \frac{2\alpha+1}{n} < \alpha^2 + (2 - \alpha^2) = 2.$$

Thus,  $\alpha + \frac{1}{n} \in T$ , contradicting the fact that  $\alpha$  is an upper bound for  $T$ . We conclude that  $\alpha^2 < 2$  cannot happen.

For the remainder of the argument, **HW:** show that  $\alpha^2 > 2$  leads to a contradiction.  $\square$

## 4 September 11, 2019

### 4.1 Cardinality

Two sets have the same cardinality if there exists a one-to-one correspondence between them.

**Definition 10.** A function  $f : A \rightarrow B$  is one-to-one (an injection) if  $a_1 \neq a_2$  in  $A$  implies  $f(a_1) \neq f(a_2)$ . The function  $f : A \rightarrow B$  is onto (a surjection) if for each  $b \in B$ ,  $\exists$  some  $a \in A$  s.t.  $f(a) = b$ .

**Definition 11.** Two sets  $A$  and  $B$  have the same cardinality if there exists a function  $f : A \rightarrow B$  that is one-to-one and onto (a bijection). In this case, we write  $A \sim B$ .

#### Example: 4.1.1

If we let  $A = \{2, 4, 6, \dots\}$  be the set of even natural numbers, then we can show  $\mathbb{N} \sim A$ .

#### Example: 4.1.2

$\mathbb{N} \sim \mathbb{Z}$ .

*Proof.* Let

$$f(n) = \begin{cases} -\frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

The important details to verify are that  $f$  does not map any two natural numbers to the same element of  $\mathbb{Z}$  and that every element of  $\mathbb{Z}$  gets “hit” by something in  $\mathbb{N}$ .  $\square$

**Definition 12.** A set  $A$  is countable if  $\mathbb{N} \sim A$ . An infinite set that is not countable is called uncountable.

#### Theorem: 4.1.1

1.  $\mathbb{Q}$  is countable.
2.  $\mathbb{R}$  is uncountable.

*Proof.* For part (1), construct  $f : \mathbb{N} \rightarrow \mathbb{Q}$  and set  $A_1 = \{0\}$ . For each  $n \geq 2$ , let the set  $A_n$  be given by

$$A_n = \left\{ \pm \frac{p}{q} \mid p, q \in \mathbb{N}, \gcd(p, q) = 1, p + q = n \right\}.$$

Then, the first few  $A_n$  are

$$A_2 = \left\{ \frac{1}{1}, -\frac{1}{1} \right\}, \quad A_3 = \left\{ \frac{2}{1}, -\frac{2}{1}, \frac{1}{2}, -\frac{1}{2} \right\}, \quad A_4 = \left\{ \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3} \right\}.$$

Each  $A_n$  is finite and every rational number in lowest terms appears in exactly one of these sets. A one-to-one correspondence with  $\mathbb{N}$  is achieved by consecutively listing the elements in each  $A_n$ .

For part (2), argue by contradiction. Assume  $\mathbb{R}$  is countable and write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\}.$$

We will use the Nested Interval Property (NIP) to produce a real number not on this list. Let  $I_1$  be a closed interval not containing  $x_1$ , and let  $I_2 \subseteq I_1$  be a closed interval with  $x_2 \notin I_2$ . In general, pick  $I_n \subseteq I_{n-1}$  so that  $x_n \notin I_n$ . Then we have constructed a sequence of nested intervals  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$  with the property that no  $x_i$  lies in  $\bigcap_{i=1}^{\infty} I_i$  (since  $x_i \notin I_i$  for each  $i$ ). However, NIP says that  $\bigcap_{i=1}^{\infty} I_i \neq \emptyset$ . Thus there exists  $x \in \bigcap_{i=1}^{\infty} I_i$  that is not on the list  $\{x_1, x_2, \dots\}$ . This contradiction shows that  $\mathbb{R}$  is uncountable.  $\square$

## 5 September 13, 2019

### 5.1 The Limit of a Sequence

**Definition 13.** A sequence is a function whose domain is  $\mathbb{N}$ ; i.e., an ordered list of numbers (objects).

#### Example: 5.1.1

1.  $\{a_1, a_2, a_3, \dots\}$ .
2. For  $f : \mathbb{N} \rightarrow \mathbb{R}$ ,  $f(1) = a_1, f(2) = a_2, \dots$

**Definition 14. (Convergence of a Sequence).** A sequence  $(a_n)$  converges to a real number  $a$  if for each  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - a| < \epsilon$ .

If a sequence does not converge, we say it *diverges*. To indicate that  $(a_n)$  converges to  $a$ , we usually write  $\lim_{n \rightarrow \infty} a_n = a$ .

**Definition 15.** Given  $a \in \mathbb{R}$  and a number  $\epsilon > 0$ , the  $\epsilon$ -neighborhood of  $a$  is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \epsilon\}.$$

**Definition 16. (Convergence of a Sequence: Topological Version).** A sequence  $(a_n)$  converges to  $a$  if, given any  $\epsilon$ -neighborhood of  $a$ , there exists a point in the sequence after which all of the terms are in  $V_\epsilon(a)$ . In other words, every  $\epsilon$ -neighborhood contains all but finitely many terms of the sequence.

#### Theorem: 5.1.1

**(Uniqueness of Limits).** The limit of a sequence, when it exists, is unique.

## 6 September 18, 2019

### 6.1 Algebraic Limit Theorem

**Definition 17.** A sequence  $(x_n)$  is bounded if there exists a number  $M > 0$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

#### Theorem: 6.1.1

Every convergent sequence is bounded.

*Proof.* Assume  $(x_n)$  converges to a limit  $l$ . This means that given a particular value of  $\epsilon$ , say  $\epsilon = 1$ , there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $x_n \in (l - 1, l + 1)$ . Not knowing whether  $l$  is positive or negative, we can certainly conclude that

$$|x_n| < |l| + 1$$

for all  $n \geq N$ .

Because there are only finitely many terms before  $N$ , let

$$M = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_{N-1}|, |l| + 1\}.$$

It follows that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ , as desired.  $\square$

#### Theorem: 6.1.2

**(Algebraic Limit Theorem).** Let  $\lim a_n = a$  and  $\lim b_n = b$ . Then

1.  $\lim(ca_n) = ca$ , for all  $c \in \mathbb{R}$ ;
2.  $\lim(a_n + b_n) = a + b$ ;
3.  $\lim(a_n b_n) = ab$ ;
4.  $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ , provided  $b \neq 0$  (and  $b_n \neq 0$  for all sufficiently large  $n$ ).

## 6.2 Order Limit Theorem

### Theorem: 6.2.1

**(Order Limit Theorem).** Assume  $(a_n) \rightarrow a$  and  $(b_n) \rightarrow b$ . Then

1. If  $a_n \geq 0$  for all  $n \in \mathbb{N}$ , then  $a \geq 0$ .
2. If  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then  $a \leq b$ .
3. If  $\exists c \in \mathbb{R}$  such that  $c \leq b_n$  for all  $n \in \mathbb{N}$ , then  $c \leq b$ .

*Proof.* For part (1), assume  $a < 0$  and let  $\epsilon \leq |a|$ . Since  $(a_n) \rightarrow a$ ,  $\exists N \in \mathbb{N}$  such that if  $n \geq N$ , then  $|a_n - a| < \epsilon$ ; equivalently,  $a_n \in V_\epsilon(a)$  for all  $n \geq N$ , which implies  $a_n < 0$  in particular for  $n = N$ . This contradicts our hypothesis that  $a_N \geq 0$ . Therefore  $a \geq 0$ .

For part (2), the Algebraic Limit Theorem ensures that the sequence  $(b_n - a_n)$  converges to  $b - a$ . Assuming  $b_n \geq a_n$ , we have  $b_n - a_n \geq 0$ . By part (1),  $b - a \geq 0$ , i.e.,  $a \leq b$ .

For part (3), assume  $\exists c$  s.t.  $c \leq b_n$  for all  $n$ . Define the constant sequence  $c_n \equiv c$ , so  $c_n \rightarrow c$ . Since  $c_n \leq b_n$  for all  $n \in \mathbb{N}$ , part (2) gives  $c \leq b$ .  $\square$

## 7 September 20, 2019

### 7.1 The Monotone Convergence Theorem

**Definition 18.** A sequence is increasing if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$  and decreasing if  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence is monotone if it is increasing or decreasing.

#### Theorem: 7.1.1

**Monotone Convergence Theorem (MCT).** A bounded and monotone sequence converges.

*Proof.* Let  $(a_n)$  be bounded and monotone. Assume the sequence is increasing, so  $a_n \leq a_{n+1}$ . By the Axiom of Completeness,  $\alpha = \sup\{a_n : n \in \mathbb{N}\}$  exists.

Let  $\epsilon > 0$  be given. Since  $\alpha = \sup(a_n)$ , there exists  $N$  such that  $\alpha - \epsilon < a_N$ . Because  $(a_n)$  is increasing, if  $n \geq N$  then  $a_N \leq a_n \leq \alpha$ . Hence

$$\alpha - \epsilon < a_N \leq a_n \leq \alpha < \alpha + \epsilon,$$

which implies  $|a_n - \alpha| < \epsilon$ , as desired.  $\square$

### 7.2 A First Look at Infinite Series

**Definition 19. (Convergence of a Series).** Let  $(b_n)$  be a sequence. An infinite series is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots$$

We define the corresponding sequence of partial sums  $(s_m)$  by

$$s_m = b_1 + b_2 + \cdots + b_m,$$

and say that the series  $\sum_{n=1}^{\infty} b_n$  converges to  $B$  if the sequence  $(s_m)$  converges to  $B$ . In this case, we write  $\sum_{n=1}^{\infty} b_n = B$ .

#### Example: 7.2.1

The series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges.

*Proof.* Take a few terms of the series,

$$s_1 = 1, \quad s_2 = 1 + \frac{1}{4}, \quad s_3 = 1 + \frac{1}{4} + \frac{1}{9}, \quad \text{and } s_m \leq s_{m+1}.$$

We note that the sequence of partial sums is increasing. To show convergence, it suffices to show  $(s_m)$  is bounded. For  $m \geq 2$ ,

$$\begin{aligned} s_m &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{m^2} \\ &\leq 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \cdots + \frac{1}{m(m-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= 2 - \frac{1}{m} < 2. \end{aligned}$$

Thus  $(s_m)$  is increasing and bounded, so by the Monotone Convergence Theorem,  $(s_m)$  converges.  $\square$

### Example: 7.2.2

The series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

*Proof.* Similarly, we have an increasing sequence of partial sums,

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Consider

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 + \frac{1}{2}, \\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}, \\ s_8 &= 1 + \frac{1}{2} + \cdots + \frac{1}{8} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}, \dots \end{aligned}$$

In general,  $s_{2^k} \geq 1 + \frac{1}{2} + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{k-1 \text{ times}} = 1 + \frac{k}{2}$ . Hence the partial sums are unbounded.

Because convergent sequences are bounded, the harmonic series diverges.  $\square$

**Theorem: 7.2.1**

**(Cauchy Condensation Test).** Let  $(b_n)$  be a decreasing sequence with  $b_n \geq 0$  for all  $n \in \mathbb{N}$ . Then  $\sum_{n=1}^{\infty} b_n$  converges iff

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + \dots$$

converges.

*Proof.* First, assume that  $\sum_{n=0}^{\infty} 2^n b_{2^n}$  converges. Then its partial sums

$$t_k = b_1 + 2b_2 + 2^2b_4 + 2^3b_8 + \dots + 2^k b_{2^k}$$

are bounded: there exists  $M > 0$  such that  $t_k \leq M$  for all  $k$ . Consider the partial sums of  $\sum b_n$ ,

$$s_m = b_1 + b_2 + \dots + b_m.$$

Since  $(s_m)$  is increasing (because  $b_n \geq 0$ ), it suffices to show that  $(s_m)$  is bounded. For  $m \leq 2^{k+1} - 1$  (with  $k$  large enough),

$$\begin{aligned} s_m &\leq s_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \dots + (b_{2^k} + \dots + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \dots + \underbrace{(b_{2^k} + \dots + b_{2^k})}_{2^k \text{ terms}} \\ &= b_1 + 2b_2 + 4b_4 + \dots + 2^k b_{2^k} = t_k \leq M. \end{aligned}$$

Thus  $(s_m)$  is bounded, and by the Monotone Convergence Theorem,  $\sum_{n=1}^{\infty} b_n$  converges. The converse direction is proved similarly by grouping terms and using the monotonicity of  $(b_n)$ .  $\square$

## 8 September 23, 2019

### 8.1 Subsequences

**Definition 20.** Let  $(a_n)$  be a sequence and let  $n_1 < n_2 < n_3 < \dots$  be an increasing sequence of natural numbers. The sequence  $(a_{n_k}) = (a_{n_1}, a_{n_2}, a_{n_3}, \dots)$  is a subsequence of  $(a_n)$ , where  $k$  indexes the subsequence.

#### Theorem: 8.1.1

Subsequences of a convergent sequence converge to the same limit as the original sequence.

*Proof.* Assume  $(a_n) \rightarrow a$  and let  $(a_{n_k})$  be a subsequence. Given  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $|a_n - a| < \epsilon$  for all  $n \geq N$ . Because  $n_k \geq k$  for all  $k$ , if  $k \geq N$  then  $n_k \geq N$ , and hence  $|a_{n_k} - a| < \epsilon$  for all  $k \geq N$ .  $\square$

This theorem is used as a *divergence criterion*.

#### Example: 8.1.1

Consider the sequence

$$(a_n) = ((-1)^n) = (-1, 1, -1, 1, \dots).$$

The subsequences  $(a_{2k}) = (1, 1, 1, \dots)$  and  $(a_{2k-1}) = (-1, -1, -1, \dots)$  have different limits. Thus, by **Theorem 8.1.1**,  $(a_n)$  diverges.

### 8.2 The Bolzano-Weierstrass Theorem

#### Theorem: 8.2.1

**(Bolzano–Weierstrass Theorem).** Every bounded sequence has a convergent subsequence.

*Proof.* Let  $(a_n)$  be a bounded sequence; i.e., there exists  $M > 0$  with  $|a_n| \leq M$  for all  $n \in \mathbb{N}$ . Bisect the closed interval  $[-M, M]$  into  $[-M, 0]$  and  $[0, M]$ . At least one of these contains infinitely many terms of  $(a_n)$ .

Let  $I_1$  be one such interval and pick  $a_{n_1} \in I_1$ . Bisect  $I_1$  into two equal closed subintervals; again one contains infinitely many terms of  $(a_n)$ . Call this subinterval  $I_2$  and pick  $a_{n_2} \in I_2$  with  $n_2 > n_1$ . In general, construct  $I_k \subseteq I_{k-1}$  by halving  $I_{k-1}$  and selecting the half

containing infinitely many terms of  $(a_n)$ ; then choose  $n_k > n_{k-1}$  with  $a_{n_k} \in I_k$ . The intervals form a nested sequence

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots,$$

so by the NIP,  $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$ . Let  $x \in \bigcap_{k=1}^{\infty} I_k$ . The length of  $I_k$  is  $M/2^{k-1}$ , so given  $\epsilon > 0$  choose  $N$  with  $M/2^{N-1} < \epsilon$ . For  $k \geq N$ ,  $x, a_{n_k} \in I_k$  implies  $|a_{n_k} - x| < \epsilon$ . Hence  $(a_{n_k}) \rightarrow x$ .  $\square$

## 9 September 25, 2019

### 9.1 The Cauchy Criterion

**Definition 21.** A sequence  $(a_n)$  is called a Cauchy sequence if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that if  $m, n \geq N$ , it follows that  $|a_n - a_m| < \epsilon$ .

To make the comparison easier, let us restate the definition of convergence.

**Remark 22.** A sequence  $(a_n)$  converges to  $a$  if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $|a_n - a| < \epsilon$ .

Thus, convergent sequences are Cauchy, and (in  $\mathbb{R}$ ) Cauchy sequences converge.

**Lemma 23.** Cauchy sequences are bounded.

*Proof.* Let  $(a_n)$  be Cauchy and take  $\epsilon = 1$ . Then there exists  $N$  such that  $|a_m - a_n| < 1$  for all  $m, n \geq N$ . This implies  $|a_n| \leq |a_N| + 1$  for all  $n \geq N$ . It follows that

$$M = \max\{|a_1|, |a_2|, |a_3|, \dots, |a_{N-1}|, |a_N| + 1\}$$

is a bound for the sequence  $(a_n)$ . □

#### Theorem: 9.1.1

A sequence  $(a_n)$  converges if and only if  $(a_n)$  is a Cauchy sequence.

*Proof.* ( $\implies$ ) Suppose  $(a_n)$  converges to  $a$  and let  $\epsilon > 0$  be given. Choose  $N \in \mathbb{N}$  so that  $|a_n - a| < \frac{\epsilon}{2}$  for all  $n \geq N$ . For  $m, n \geq N$ ,

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leq |a_n - a| + |a_m - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

( $\impliedby$ ) Let  $(a_n)$  be Cauchy. By the lemma above,  $(a_n)$  is bounded, so by the Bolzano–Weierstrass Theorem there exists a convergent subsequence  $(a_{n_k}) \rightarrow a$ . Let  $\epsilon > 0$  be given. Choose  $N$  so that  $|a_n - a_m| < \frac{\epsilon}{2}$  for all  $n, m \geq N$ . Also, because  $(a_{n_k}) \rightarrow a$ , pick  $K$  with  $n_K \geq N$  and  $|a_{n_K} - a| < \frac{\epsilon}{2}$ . Then for any  $n \geq N$ ,

$$\begin{aligned} |a_n - a| &= |a_n - a_{n_K} + a_{n_K} - a| \\ &\leq |a_n - a_{n_K}| + |a_{n_K} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence  $(a_n) \rightarrow a$ . □

## 10 September 27, 2019

### 10.1 Metric Spaces

Interpret  $|x - y|$  to mean the *distance* from  $x$  to  $y$  in  $\mathbb{R}$ ; it has the following key features:

- $|x - y| \geq 0$ .
- $|x - y| = 0 \iff x = y$ .
- $|x - y| = |y - x|$ .
- $|x - y| \leq |x - z| + |y - z|$  (*Triangle Inequality*).

**Definition 24.** A metric on a set  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  if for all  $x, y, z \in X$ :

1.  $d(x, y) \geq 0$  with  $d(x, y) = 0$  iff  $x = y$  (*positive definiteness*),
2.  $d(x, y) = d(y, x)$  (*symmetry*), and
3.  $d(x, y) \leq d(x, z) + d(z, y)$  (*triangle inequality*).

A metric space is a set  $X$  equipped with a metric  $d$ .

#### Example: 10.1.1

**(Taxi Cab Metric).** On  $X = \mathbb{R}^2$ , let  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  be points in the plane. Verify

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

is a metric.

(We only check (3).) For any  $z = (z_1, z_2)$ ,

$$\begin{aligned} d(x, y) &= |x_1 - y_1| + |x_2 - y_2| \\ &= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2| \\ &\leq |x_1 - z_1| + |x_2 - z_2| + |z_1 - y_1| + |z_2 - y_2| \\ &= d(x, z) + d(z, y). \end{aligned}$$

#### Example: 10.1.2

**(Discrete Metric).** Let  $X$  be any set. For  $x, y \in X$  define

$$\rho(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

## 10.2 Basis Definitions

**Definition 25.** Let  $(X, d)$  be a metric space. A sequence  $(x_n)$  converges to  $x \in X$  if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for  $n \geq N$ ,  $d(x_n, x) < \epsilon$ .

**Definition 26.** A sequence  $(x_n)$  in a metric space  $(X, d)$  is Cauchy if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that if  $m, n \geq N$ , then  $d(x_n, x_m) < \epsilon$ .

**Definition 27.** A metric space  $(X, d)$  is complete if every Cauchy sequence in  $X$  converges to an element of  $X$ .

## 11 September 30, 2019

### 11.1 Discussion: The Cantor Set

Let  $C_0$  be the closed interval  $[0, 1]$ , and define  $C_1$  to be the set that results when the open middle third is removed; that is,

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now, construct  $C_2$  in a similar way by removing the open middle third of each of the two components of  $C_1$ :

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right).$$

If we continue this process inductively, then for each  $n = 0, 1, 2, \dots$  we get a set  $C_n$  consisting of  $2^n$  closed intervals each having length  $\frac{1}{3^n}$ .

**Definition 28.** *The Cantor set  $C$  is the intersection*

$$C = \bigcap_{k=0}^{\infty} C_k.$$

At stage  $n$  we remove  $2^{n-1}$  middle thirds of length  $\frac{1}{3^n}$ . Thus, the total length of  $C_n$  is  $\left(\frac{2}{3}\right)^n$ . The Cantor set has *zero length*.

## 12 October 2/4, 2019

### 12.1 Open and Closed Sets

Given  $a \in \mathbb{R}$  and  $\epsilon > 0$ , recall that the  $\epsilon$ -neighborhood of a point  $a$  is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon),$$

centered at  $a$  with radius  $\epsilon$ .

**Definition 29.** A set  $O \subseteq \mathbb{R}$  is open if for all  $a \in O$ ,  $\exists$  an  $\epsilon$ -neighborhood  $V_\epsilon(a) \subseteq O$ .

Note: consider the open interval

$$(c, d) = \{x \in \mathbb{R} \mid c < x < d\},$$

let  $x \in (c, d)$  be arbitrary. If we take  $\epsilon = \min\{d-x, x-c\}$ , then it follows that  $V_\epsilon(x) \subseteq (c, d)$ .

#### Theorem

1. The union of an arbitrary collection of open sets is open.
2. The intersection of a finite collection of open sets is open.

*Proof.* To prove (1), let  $\{A_k\}_{k \in K}$  be a collection of open sets and let  $x \in \bigcup_{k \in K} A_k$ . Then  $x \in A_j$  for some  $j$ . By the definition of open, there exists  $V_\epsilon(x) \subseteq A_j$ . Hence  $V_\epsilon(x) \subseteq \bigcup_{k \in K} A_k$ .

For (2), let  $A_1, \dots, A_n$  be a finite collection of open sets. If  $x \in \bigcap_{k=1}^n A_k$ , then  $x$  is an element of each open set. For each  $1 \leq k \leq n$ , there exists  $V_{\epsilon_k}(x) \subseteq A_k$ . Letting  $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$ , it follows that  $V_\epsilon(x) \subseteq V_{\epsilon_k}(x)$  for any  $k$ , and hence  $V_\epsilon(x) \subseteq \bigcap_{k=1}^n A_k$ , as desired.  $\square$

## 13 October 7, 2019

### 13.1 Compact Sets

**Definition 30. (Compactness).** A set  $K \subseteq \mathbb{R}$  is compact if every sequence in  $K$  has a subsequence that converges to a limit that is also in  $K$ .

**Remark 31. Heuristics:**

1. “Every sequence has a convergent subsequence” (Bolzano–Weierstrass for bounded sequences) accounts for boundedness of  $K$ .
2. “The subsequential limit lies in  $K$ ” corresponds to closedness of  $K$ .

**Definition 32.** A set  $A \subseteq \mathbb{R}$  is bounded if there exists  $M > 0$  such that  $|a| \leq M$  for all  $a \in A$ .

#### Theorem: 13.1.1

**(Characterization of Compactness in  $\mathbb{R}$ ).** A set  $K \subseteq \mathbb{R}$  is compact if and only if it is closed and bounded.

*Proof.* ( $\implies$ ) Assume  $K$  is compact. First,  $K$  is bounded: otherwise, for each  $n \in \mathbb{N}$  there exists  $x_n \in K$  with  $|x_n| > n$ . By compactness,  $(x_n)$  has a convergent subsequence  $(x_{n_k}) \rightarrow x \in K$ , but then  $(x_{n_k})$  is bounded, a contradiction.

Next,  $K$  is closed: let  $(x_n) \subseteq K$  with  $x_n \rightarrow x$ . By compactness,  $(x_n)$  has a convergent subsequence with limit in  $K$ ; uniqueness of limits gives  $x \in K$ .

( $\impliedby$ ) The converse is assigned in HW. □

#### Theorem: 13.1.2

**(Nested Compact Set Property).** If

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

is a nested sequence of nonempty compact sets, then the intersection  $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$ .

*Proof.* Pick  $x_k \in I_k$  for each  $k \in \mathbb{N}$ . Because the sets are nested,  $(x_k) \subseteq I_1$ . By compactness of  $I_1$ , there is a convergent subsequence  $(x_{k_p})$  with limit  $x \in I_1$ .

Given  $k_0 \in \mathbb{N}$ , all terms  $x_k$  with  $k \geq k_0$  lie in  $I_{k_0}$ ; hence the tail of  $(x_{k_p})$  lies in  $I_{k_0}$ . Since each  $I_{k_0}$  is compact (in particular, closed), the limit  $x$  belongs to  $I_{k_0}$ . As  $k_0$  was arbitrary,  $x \in \bigcap_{k=1}^{\infty} I_k$ . □

## 14 October 9, 2019

### 14.1 Open Covers

There is another important characterization of compactness, equivalent to **Theorem 13.1.1**, which is described in terms of *open covers* and *finite subcovers*.

**Definition 33.** Let  $A \subseteq \mathbb{R}$ . An *open cover* of  $A$  is a collection of open sets  $\{O_\alpha \mid \alpha \in J\}$  whose union contains the set  $A$ ; that is,  $A \subseteq \bigcup_{\alpha \in J} O_\alpha$ .

**Definition 34.** Given an open cover of  $A$ , a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still completely contains  $A$ .

#### Example: 14.1.1

Consider the open interval  $A = (0, 1)$ . For each  $t \in (0, 1)$ , let  $O_t = (t/2, 1)$ . Taken together,  $\{O_t \mid t \in (0, 1)\}$  forms an open cover for  $(0, 1)$ . It is impossible to find a finite subcover: given any finite subcollection

$$\{O_{t_1}, O_{t_2}, \dots, O_{t_m}\},$$

set  $t' = \min\{t_1, t_2, \dots, t_m\}$  and observe that any  $y$  with  $0 < y \leq t'/2$  is not contained in  $\bigcup_{i=1}^m O_{t_i}$ .

Now consider a similar cover for the closed interval  $[0, 1]$ . Fix  $\epsilon > 0$ , and let  $O_0 = (-\epsilon, \epsilon)$  and  $O_1 = (1 - \epsilon, 1 + \epsilon)$ . Then the collection

$$\{O_0, O_1\} \cup \{O_t : t \in (0, 1)\}$$

is an open cover for  $[0, 1]$ . Because of the addition of  $O_0$ , choose  $t'$  so that  $t'/2 < \epsilon$ . It follows that  $\{O_0, O_{t'}, O_1\}$  is a finite subcover for  $[0, 1]$ .

#### Theorem: 14.1.1

**(Heine–Borel Theorem).** Let  $K \subseteq \mathbb{R}$ . The following are equivalent (*TFAE*):

1. Every sequence  $(a_n) \subseteq K$  has a convergent subsequence  $(a_{n_k})$  whose limit is in  $K$  (*Sequential Compactness*).
2.  $K$  is closed and bounded.
3. Every open cover of  $K$  has a finite subcover (*Topological Compactness*).

Note: we have shown (1) and (2) are equivalent in the last lecture.

## 15 October 11/16, 2019

### 15.1 Perfect Sets

**Definition 35.** A set  $P \subseteq \mathbb{R}$  is **perfect** if it is closed and contains no isolated points.

#### Example: 15.1.1

Define the Cantor set as

$$C = \bigcap_{n=0}^{\infty} C_n,$$

where each  $C_n$  is a finite union of closed intervals.

#### Theorem: 15.1.1

A nonempty perfect set is uncountable.

### 15.2 Connected Sets

**Definition 36.** Two nonempty sets  $A, B \subseteq \mathbb{R}$  are **separated** if  $\bar{A} \cap B = \emptyset$  and  $A \cap \bar{B} = \emptyset$ . A set  $E \subseteq \mathbb{R}$  is **disconnected** if it can be written as  $E = A \cup B$ , where  $A$  and  $B$  are nonempty separated sets.

A set that is not disconnected is called a **connected set**.

#### Example: 15.2.1

Let  $A = (1, 2)$  and  $B = (2, 5)$ . Then

$$E = (1, 2) \cup (2, 5)$$

is *disconnected*.

#### Theorem: 15.2.1

A set  $E \subseteq \mathbb{R}$  is connected if and only if, for all nonempty disjoint sets  $A$  and  $B$  satisfying  $E = A \cup B$ , there exists a convergent sequence  $(x_n) \rightarrow x$  with  $(x_n)$  contained in one of  $A$  or  $B$ , and  $x$  an element of the other.

#### Theorem: 15.2.2

A set  $E \subseteq \mathbb{R}$  is connected if and only if whenever  $a < c < b$  with  $a, b \in E$ , it follows that  $c \in E$  as well.

### 15.3 Nowhere-Dense Sets

**Definition 37.** A set  $E$  is nowhere-dense if  $\overline{E}$  contains no nonempty open intervals.

**Theorem: 15.3.1**

**(Baire's Theorem).** The set of real numbers  $\mathbb{R}$  cannot be written as the countable union of nowhere-dense sets.

## 16 October 18, 2019

### 16.1 Discussion: Examples of Dirichlet and Thomae

What does “ $\lim_{x \rightarrow a} f(x) = L$ ” mean?

“ $f(x)$  gets close to  $L$  as  $x$  gets close to  $a$ .” However, this statement says nothing about whether  $f(a)$  is even defined.

#### Example: 17.1.1

(Dirichlet’s Function) Let

$$g(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Question: what is  $\lim_{x \rightarrow 0} g(x)$ ? 1? 0? Limit DNE?

*Solution.* Let  $x_n = \frac{1}{n}$  (rational). Then  $\lim_{n \rightarrow \infty} g(x_n) = 1$ . On the other hand, let  $x_n = \frac{\sqrt{2}}{n}$  (irrational). Then  $\lim_{n \rightarrow \infty} g(x_n) = 0$ .

Thus  $g$  is not continuous at 0. Generally, we want  $\lim_{x \rightarrow c} g(x)$  to be independent of how we approach  $c$ . Here, for any  $c \in \mathbb{R}$ ,  $\lim_{x \rightarrow c} g(x)$  does not exist;  $g$  is *nowhere continuous*. ■

#### Example: 17.1.2

Define  $h : \mathbb{R} \rightarrow \mathbb{R}$  by

$$h(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

*Solution.* If  $c \neq 0$ , pick sequences of rationals  $(x_n) \rightarrow c$  and irrationals  $(y_n) \rightarrow c$ , then

$$\lim_{n \rightarrow \infty} h(x_n) = c \quad \text{and} \quad \lim_{n \rightarrow \infty} h(y_n) = 0.$$

If  $c = 0$ , every such choice gives limit 0. Hence  $h$  is continuous only at 0. ■

### 16.2 Functional Limits

**Definition 38. (Functional Limit).** Let  $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , and let  $c$  be a limit point of  $A$ . We say that  $\lim_{x \rightarrow c} f(x) = L$  if for each  $\epsilon > 0$  there exists  $\delta > 0$  such that for all  $x \in A$  with  $0 < |x - c| < \delta$ , we have  $|f(x) - L| < \epsilon$ .

This is often referred to as the “ $\epsilon$ - $\delta$  version” of the definition for functional limits.

**Definition 39. (Functional Limit: Topological Version).** Let  $f : A \rightarrow \mathbb{R}$ , and let  $c$  be a limit point of  $A$ . We say  $\lim_{x \rightarrow c} f(x) = L$  if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in V_\delta(c) \cap A$  with  $0 < |x - c| < \delta$ , it follows that  $f(x) \in V_\epsilon(L)$ .

*Challenge Response:*

Given  $\epsilon > 0$ , find  $\delta > 0$  so that if  $0 < |x - c| < \delta$  then  $|f(x) - L| < \epsilon$ ; in other words,

$$f(V_\delta(c) \cap A) \subseteq V_\epsilon(L).$$

#### Example: 17.2.1

Let's prove that if  $f(x) = 2x - 4$ , then

$$\lim_{x \rightarrow 1} f(x) = -2.$$

*Solution.* Given  $\epsilon > 0$ , we want  $\delta > 0$  so that  $0 < |x - 1| < \delta$  implies  $|f(x) - (-2)| < \epsilon$ . Note

$$|f(x) - (-2)| = |2x - 4 + 2| = |2x - 2| = 2|x - 1|.$$

Thus, choosing  $\delta = \epsilon/2$  gives  $|f(x) - (-2)| < 2(\epsilon/2) = \epsilon$ . ■

#### Example: 17.2.2

Given

$$\lim_{x \rightarrow 3} x^2 = 9,$$

find a  $\delta$  so that  $|x^2 - 9| < \epsilon$  whenever  $|x - 3| < \delta$ .

*Solution.* We have

$$|x^2 - 9| = |x - 3||x + 3|.$$

If we additionally require  $\delta \leq 1$ , then  $|x - 3| < 1$  implies  $2 < x < 4$ , hence  $|x + 3| \leq 7$ . Therefore

$$|x^2 - 9| = |x - 3||x + 3| < \delta \cdot 7.$$

Choose  $\delta = \min\{1, \epsilon/7\}$ . Then  $|x^2 - 9| < \epsilon$ . ■

## 17 October 21/23, 2019

**Recall:**  $\lim_{x \rightarrow c} f(x) = L$  means for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $0 < |x - c| < \delta$  implies  $|f(x) - L| < \epsilon$ .

Topological version:

For each  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $x \in V_\delta(c) \cap A$  with  $x \neq c$ , then it follows that  $f(x) \in V_\epsilon(L)$ .

### 17.1 Sequential Criterion for Functional Limits

#### Theorem: 18.1.1

**(Sequential Characterization for Functional Limits).** Given  $f : A \rightarrow \mathbb{R}$  and a limit point  $c$  of  $A$ , the following are equivalent (TFAE):

1.  $\lim_{x \rightarrow c} f(x) = L$ ;
2. for every sequence  $(x_n) \subseteq A$  with  $x_n \neq c$  and  $x_n \rightarrow c$ , we have  $f(x_n) \rightarrow L$ .

*Proof.* **((1) $\implies$ (2))** Assume  $\lim_{x \rightarrow c} f(x) = L$ , and let  $(x_n) \subseteq A$  with  $x_n \neq c$  and  $x_n \rightarrow c$ . We want to show  $f(x_n) \rightarrow L$ . Given  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $x \in A$  and  $0 < |x - c| < \delta$ , we have  $|f(x) - L| < \epsilon$ . Since  $x_n \rightarrow c$ , there exists  $N$  with  $|x_n - c| < \delta$  for all  $n \geq N$ . Hence  $|f(x_n) - L| < \epsilon$  for all  $n \geq N$ , so  $f(x_n) \rightarrow L$ .

**((2) $\implies$ (1))** We prove the contrapositive. If (1) fails, then there exists  $\epsilon_0 > 0$  such that for every  $\delta > 0$  there is some  $x \in A$  with  $0 < |x - c| < \delta$  and  $|f(x) - L| > \epsilon_0$ . Taking  $\delta = \frac{1}{n}$  and choosing  $x_n \in A$  with  $0 < |x_n - c| < \frac{1}{n}$  and  $|f(x_n) - L| > \epsilon_0$ , we get  $x_n \rightarrow c$  but  $f(x_n) \not\rightarrow L$ . Thus (2) fails.  $\square$

**Theorem 18.1.1** has several useful corollaries.

**Corollary 40. (Algebraic Limit Theorem for Functional Limits).** Let  $f, g : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$  with  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$ . Then

1.  $\lim_{x \rightarrow c} k f(x) = kL$  for all  $k \in \mathbb{R}$ ;
2.  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$ ;
3.  $\lim_{x \rightarrow c} (f(x)g(x)) = LM$ ;
4.  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}$ , if  $M \neq 0$ .

**Corollary 41. (Divergence Criterion).** Let  $f : A \rightarrow \mathbb{R}$  and  $c$  be a limit point of  $A$ . If there exist sequences  $(x_n), (y_n) \subseteq A$  with  $x_n \neq c$ ,  $y_n \neq c$ ,  $x_n \rightarrow c$ ,  $y_n \rightarrow c$ , and  $\lim f(x_n) \neq \lim f(y_n)$  (both limits exist), then  $\lim_{x \rightarrow c} f(x)$  does not exist.

**Example: 18.1.1**

**(Heaviside Function).** Define

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Show that  $\lim_{x \rightarrow 0} H(x)$  does not exist.

*Solution.* If  $x_n \rightarrow 0$  with  $x_n > 0$ , then  $H(x_n) \rightarrow 1$ . If  $y_n \rightarrow 0$  with  $y_n < 0$ , then  $H(y_n) \rightarrow 0$ . Hence the limit does not exist. ■

**Example: 18.1.2**

Show that  $\lim_{x \rightarrow 0} \sin(1/x)$  does not exist.

*Solution.* Let  $x_n = \frac{1}{2n\pi}$  and  $y_n = \frac{1}{2n\pi + \pi/2}$ . Then  $x_n \rightarrow 0$  and  $y_n \rightarrow 0$ , while  $\sin(1/x_n) = 0$  for all  $n$  and  $\sin(1/y_n) = 1$  for all  $n$ . Thus  $\lim \sin(1/x_n) \neq \lim \sin(1/y_n)$ , so the limit at 0 does not exist. ■

**17.2 Continuous Functions**

**Definition 42. (Continuity).** A function  $f : A \rightarrow \mathbb{R}$  is continuous at  $c \in A$  if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $x \in A$  and  $|x - c| < \delta$  imply  $|f(x) - f(c)| < \epsilon$ .

$f$  is continuous at every point of  $A \iff f$  is continuous on  $A$ .

**Theorem:**

Let  $f : A \rightarrow \mathbb{R}$  and  $c \in A$ . The following are equivalent:

1. For all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $x \in A$  and  $|x - c| < \delta$  imply  $|f(x) - f(c)| < \epsilon$ ;
2. For every  $\epsilon$ -neighborhood  $V_\epsilon(f(c))$ , there exists a  $\delta$ -neighborhood  $V_\delta(c)$  with  $V_\delta(c) \cap A \subseteq f^{-1}(V_\epsilon(f(c)))$ ;
3. For all sequences  $(x_n) \subseteq A$  with  $x_n \rightarrow c$ , we have  $f(x_n) \rightarrow f(c)$ ;
4.  $\lim_{x \rightarrow c} f(x) = f(c)$ .

**Corollary 43. (Criterion for Discontinuity).** Let  $f : A \rightarrow \mathbb{R}$  and  $c \in A$  be a limit point of  $A$ . If there exists a sequence  $(x_n) \subseteq A$  with  $x_n \rightarrow c$  but  $f(x_n) \not\rightarrow f(c)$ , then  $f$  is not continuous at  $c$ .

## 18 October 28, 2019

### 18.1 More on Continuity

#### Theorem: 19.1.1

**(Characterizations of Continuity).** A function  $f : A \rightarrow \mathbb{R}$  is continuous at  $c \in A$  iff

1. For all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $x \in A$  and  $|x - c| < \delta$  imply  $|f(x) - f(c)| < \epsilon$ ;
2. For every  $\epsilon$ -neighborhood  $V_\epsilon(f(c))$  there exists a  $\delta$ -neighborhood  $V_\delta(c)$  with  $V_\delta(c) \cap A \subseteq f^{-1}(V_\epsilon(f(c)))$ ;
3. For all sequences  $(x_n) \subseteq A$  with  $x_n \rightarrow c$ , we have  $f(x_n) \rightarrow f(c)$ .

**Corollary 44.** Let  $f : A \rightarrow \mathbb{R}$  and  $c \in A$  be a limit point of  $A$ . If there exists a sequence  $(x_n) \subseteq A$  with  $(x_n) \rightarrow c$  but  $f(x_n) \not\rightarrow f(c)$ , then  $f$  is not continuous at  $c$ .

#### Theorem: 19.1.2

**(Algebraic Continuity Theorem).** Assume  $f, g : A \rightarrow \mathbb{R}$  are continuous at  $c \in A$ . Then

1.  $kf(x)$  is continuous at  $c$  for all  $k \in \mathbb{R}$ ;
2.  $f(x) + g(x)$  is continuous at  $c$ ;
3.  $f(x)g(x)$  is continuous at  $c$ ;
4.  $\frac{f(x)}{g(x)}$  is continuous at  $c$  provided  $g(c) \neq 0$ .

#### Example: 19.1.1

Polynomials are continuous on  $\mathbb{R}$ .

*Proof.* The constant function  $f(x) = k$  is continuous since  $|f(x) - f(c)| = 0$ . The identity function  $f(x) = x$  is continuous (given  $\epsilon > 0$ , take  $\delta = \epsilon$ ). By Theorem 19.1.2, sums and products of continuous functions are continuous; hence any polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

is continuous on  $\mathbb{R}$ . Rational functions (ratios of polynomials) are continuous where defined (i.e., where the denominator is nonzero).  $\square$

**Example: 19.1.2**

Consider

$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Claim:  $g$  is continuous at 0.

*Proof.* Given  $\epsilon > 0$ , take  $\delta = \epsilon$ . If  $|x - 0| < \delta$ , then

$$|g(x) - g(0)| = |x \sin(1/x)| \leq |x| < \epsilon.$$

Thus  $g$  is continuous at 0.  $\square$

**Theorem:**

Let  $f : A \rightarrow \mathbb{R}$  and  $g : B \rightarrow \mathbb{R}$  with  $f(A) \subseteq B$ , so that  $(g \circ f)(x) = g(f(x))$  is defined on  $A$ . If  $f$  is continuous at  $c \in A$  and  $g$  is continuous at  $f(c) \in B$ , then  $g \circ f$  is continuous at  $c$ .

## 19 October 30, 2019

### Theorem: 20.0.1

**(Preservation of Compact Sets).** Let  $f : A \rightarrow \mathbb{R}$  be continuous on  $A$ . If  $K \subseteq A$  is compact, then  $f(K)$  is compact as well.

An extremely important corollary is obtained by combining this result with the observation that compact sets are bounded and contain their *suprema* and *infima*.

### 19.1 Extreme Value Theorem

#### Theorem: 20.1.1

**(Extreme Value Theorem).** If  $f : K \rightarrow \mathbb{R}$  is continuous on a compact set  $K \subseteq \mathbb{R}$ , then  $f$  attains a maximum and minimum value.

*Proof.* Because  $f(K)$  is compact, we can set  $\alpha = \sup f(K)$  and know  $\alpha \in f(K)$ . It follows that there exists  $x_1 \in K$  with  $\alpha = f(x_1)$ . The argument for the minimum value is similar.  $\square$

### 19.2 Uniform Continuity

**Definition 45. (Uniform Continuity).** A function  $f : A \rightarrow \mathbb{R}$  is uniformly continuous on  $A$  if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that for all  $x, y \in A$ ,  $|x - y| < \delta$  implies  $|f(x) - f(y)| < \epsilon$ .

#### Theorem: 20.2.1

**(Sequential Criterion for Absence of Uniform Continuity).** A function  $f : A \rightarrow \mathbb{R}$  fails to be uniformly continuous on  $A$  if and only if there exists a particular  $\epsilon_0 > 0$  and two sequences  $(x_n)$  and  $(y_n)$  in  $A$  satisfying  $|x_n - y_n| \rightarrow 0$  but  $|f(x_n) - f(y_n)| \geq \epsilon_0$ .

#### Theorem: 20.2.2

**(Uniform Continuity on Compact Sets).** A function that is continuous on a compact set  $K$  is uniformly continuous on  $K$ .

## 20 November 1, 2019

### 20.1 The Intermediate Value Theorem

#### Theorem: 21.1.1

**(Intermediate Value Theorem).** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is a continuous function, and  $f(a) \leq y \leq f(b)$  (or  $f(b) \leq y \leq f(a)$ ). Then there exists  $x \in [a, b]$  such that  $y = f(x)$ .

### 20.2 Preservation of Connected Sets

#### Theorem: 21.2.1

**(Preservation of Connected Sets).** Let  $f : G \rightarrow \mathbb{R}$  be continuous. If  $E \subseteq G$  is connected, then  $f(E)$  is connected as well.

### 20.3 The Intermediate Value Property

**Definition 46.** A function  $f$  has the *intermediate value property* on an interval  $[a, b]$  if for all  $x < y$  in  $[a, b]$  and all  $L$  between  $f(x)$  and  $f(y)$ , there exists  $c \in (x, y)$  such that  $f(c) = L$ .

#### Example: 21.3.1

If  $f$  is monotone (increasing) and has the IVP, show  $f$  is continuous.

*Solution.* TBC. ■

## 21 November 4, 2019

### 21.1 Derivatives

**Definition 47. (Differentiability).** Let  $g : A \rightarrow \mathbb{R}$ , where  $A$  is an interval. Given  $c \in A$ , the derivative of  $g$  at  $c$  is defined by

$$g'(c) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists.

We say  $g$  is differentiable at  $c$ . If  $g'$  exists at every  $c \in A$ , we say  $g$  is differentiable on  $A$ .

#### Example: 22.1.1

Consider  $f(x) = x^n$ , where  $n \in \mathbb{N}$ .

*Solution.* Recall  $x^n - c^n = (x - c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \dots + c^{n-1})$ . Thus,

$$f'(c) = \lim_{x \rightarrow c} \frac{x^n - c^n}{x - c} = n c^{n-1}.$$

■

#### Example: 22.1.2

Compute the derivative of  $f(x) = |x|$  at  $c = 0$ .

*Solution.*

$$\lim_{x \rightarrow 0} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x}$$

is 1 from the right and  $-1$  from the left, so the derivative does not exist at  $c = 0$ .

■

#### Theorem: 22.1.1

If  $g : A \rightarrow \mathbb{R}$  is differentiable at  $c \in A$ , then  $g$  is continuous at  $c$ .

*Proof.* Assume

$$g'(c) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c}$$

exists. Then

$$\lim_{x \rightarrow c} (g(x) - g(c)) = \left( \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \right) \cdot \left( \lim_{x \rightarrow c} (x - c) \right) = g'(c) \cdot 0 = 0,$$

so  $\lim_{x \rightarrow c} g(x) = g(c)$ . □

## 22 November 6, 2019

### 22.1 Combinations of Differentiable Functions

#### Example: 23.1.1

Find functions  $f$  and  $g$  that are not differentiable at 0, but  $f \circ g$  is differentiable at 0.

*Solution.* Let

$$g(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0, \end{cases} \quad f(x) = \begin{cases} |x|, & x < 0, \\ 0, & 0 \leq x \leq 1, \\ x - 1, & x > 1. \end{cases}$$

Both  $g$  and  $f$  fail to be differentiable at 0, but  $(f \circ g)(x) = f(0)$  for  $x \leq 0$  and  $f(1)$  for  $x > 0$ , and  $f(0) = f(1) = 0$ . Hence  $f \circ g$  is the constant 0 near 0, so it is differentiable at 0. ■

#### Example: 23.1.2

Exactly one of the following requests is impossible.

1. A function  $f$  not differentiable at 0 and a function  $g$  differentiable at 0, where  $fg$  is differentiable at 0.
2. A function  $f$  not differentiable at 0 and a function  $g$  differentiable at 0, where  $f + g$  is differentiable at 0.

*Solution.* 1. Possible. For instance, take  $f(x) = |x|$  (not differentiable at 0) and  $g(x) = x$  (differentiable). Then  $(fg)(x) = x|x|$  is differentiable at 0.

2. Impossible. If  $f + g$  and  $g$  are differentiable at 0, then  $f = (f + g) - g$  is also differentiable at 0, a contradiction. ■

**Theorem:**

Let  $f$  and  $g$  be differentiable at  $c \in A$ , where  $A$  is an interval. Then

1.  $(f + g)'(c) = f'(c) + g'(c)$ ,
2.  $(kf)'(c) = k f'(c)$  for all  $k \in \mathbb{R}$ ,
3.  $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$ ,
4.  $(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$ , with  $g(c) \neq 0$ .

*Proof.* For (iii), rewrite the difference quotient as

$$\frac{f(x)g(x) - f(c)g(c)}{x - c} = \frac{f(x)(g(x) - g(c))}{x - c} + \frac{g(c)(f(x) - f(c))}{x - c},$$

and pass to the limit using the limits  $f(x) \rightarrow f(c)$ ,  $g(x) \rightarrow g(c)$ , and the definitions of  $f'(c)$  and  $g'(c)$ .  $\square$

## 23 November 8, 2019

### 23.1 Combinations of Differentiable Functions, Continued

#### Theorem: 24.1.1

**(Chain Rule).** Let  $f : A \rightarrow \mathbb{R}$  and  $g : B \rightarrow \mathbb{R}$  satisfy  $f(A) \subseteq B$  so that the composition  $g \circ f$  is defined. If  $f$  is differentiable at  $c \in A$  and if  $g$  is differentiable at  $f(c) \in B$ , then  $g \circ f$  is differentiable at  $c$  with  $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$ .

### 23.2 Darboux's Theorem

#### Theorem: 24.2.1

**(Interior Extremum Theorem).** Let  $f$  be differentiable on an open interval  $(a, b)$ . If  $f$  attains a maximum value at some point  $c \in (a, b)$ , then  $f'(c) = 0$  (and similarly for a minimum).

*Proof.* Let  $x_n \uparrow c$  (so  $x_n < c$ ) and  $y_n \downarrow c$  (so  $y_n > c$ ). Because  $f(c)$  is a maximum,  $f(x_n) \leq f(c)$  and  $f(y_n) \leq f(c)$  for all  $n$ . Then

$$\frac{f(x_n) - f(c)}{x_n - c} \geq 0 \quad \text{and} \quad \frac{f(y_n) - f(c)}{y_n - c} \leq 0.$$

Taking limits gives the left derivative  $f'_-(c) \geq 0$  and the right derivative  $f'_+(c) \leq 0$ . If  $f'(c)$  exists, then  $f'_-(c) = f'_+(c) = f'(c)$ , hence  $f'(c) = 0$ .  $\square$

#### Theorem: 24.2.2

**(Darboux's Theorem).** If  $f$  is differentiable on an interval  $[a, b]$ , and if  $\alpha$  satisfies  $f'(a) < \alpha < f'(b)$  (or  $f'(a) > \alpha > f'(b)$ ), then there exists a point  $c \in (a, b)$  where  $f'(c) = \alpha$ .

## 24 November 11, 2019

### 24.1 Mean Value Theorem

#### Theorem: 24.1.1

**(Rolle's Theorem).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a) = f(b)$ , then there exists a point  $c \in (a, b)$  where  $f'(c) = 0$ .

#### Theorem: 24.1.2

**(Mean Value Theorem).** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists a point  $c \in (a, b)$  where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Corollary 48.** If  $g : A \rightarrow \mathbb{R}$  is differentiable on an interval  $A$  and satisfies  $g'(x) = 0$  for all  $x \in A$ , then  $g(x) = k$  for some constant  $k \in \mathbb{R}$ .

**Corollary 49.** If  $f$  and  $g$  are differentiable functions on an interval  $A$  and satisfy  $f'(x) = g'(x)$  for all  $x \in A$ , then  $f(x) = g(x) + k$  for some constant  $k \in \mathbb{R}$ .

#### Theorem: 24.1.3

**(Generalized Mean Value Theorem).** If  $f$  and  $g$  are continuous on the closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$ , then there exists a point  $c \in (a, b)$  where

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If  $g'$  is never zero on  $(a, b)$ , then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

## 24.2 L'Hospital's Rules

### Theorem: 24.2.1

**(L'Hospital's Rule: 0/0 case).** Let  $f$  and  $g$  be continuous on an interval containing  $a$ , and assume  $f$  and  $g$  are differentiable on this interval with the possible exception of the point  $a$ . If  $f(a) = g(a) = 0$  and  $g'(x) \neq 0$  for all  $x \neq a$ , then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

**Definition 50.** Given  $g : A \rightarrow \mathbb{R}$  and a limit point  $c$  of  $A$ , we say that  $\lim_{x \rightarrow c} g(x) = \infty$  if, for every  $M > 0$ , there exists a  $\delta > 0$  such that whenever  $x \in A$  and  $0 < |x - c| < \delta$  it follows that  $g(x) \geq M$ .

### Theorem: 24.2.2

**(L'Hospital's Rule:  $\infty/\infty$  case).** Assume  $f$  and  $g$  are differentiable on  $(a, b)$  and that  $g'(x) \neq 0$  for all  $x \in (a, b)$ . If  $\lim_{x \rightarrow a} g(x) = \infty$ , then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

## 25 November 15, 2019

It is assumed that we are working with a *bounded* function  $f : [a, b] \rightarrow \mathbb{R}$ , meaning that there exists an  $M > 0$  such that  $|f(x)| \leq M$  for all  $x \in [a, b]$ .

### 25.1 Partitions, Upper Sums, and Lower Sums

**Definition 51.** A *partition*  $P$  of  $[a, b]$  is a finite set of points in  $[a, b]$  that includes both  $a$  and  $b$ . Notationally:

$$P = \{x_0, x_1, \dots, x_n\}$$

in increasing order; thus,

$$a = x_0 < x_1 < x_2 < \dots < x_n = b.$$

On each subinterval  $[x_{k-1}, x_k]$  of  $P$ , let

$$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}, \quad M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

The *lower sum* w.r.t.  $P$  is given by

$$L(f, P) = \sum_{k=1}^n m_k(x_k - x_{k-1}).$$

Likewise, we define the *upper sum* w.r.t.  $P$  by

$$U(f, P) = \sum_{k=1}^n M_k(x_k - x_{k-1}).$$

**Definition 52.** A partition  $Q$  is a *refinement* of  $P$  if  $P \subseteq Q$ . ( $Q$  contains all the points of  $P$ .)

**Lemma 53.** If  $P \subseteq Q$ , then  $L(f, P) \leq L(f, Q)$  and  $U(f, P) \geq U(f, Q)$ .

**Lemma 54.** If  $P_1$  and  $P_2$  are any two partitions of  $[a, b]$ , then  $L(f, P_1) \leq U(f, P_2)$ .

*Proof.* Let  $Q = P_1 \cup P_2$  be the common refinement of  $P_1$  and  $P_2$ . Because  $P_1 \subseteq Q$  and  $P_2 \subseteq Q$ , it follows that

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2).$$

□

**Definition 55.** Let  $\mathcal{P}$  be the collection of all possible partitions of  $[a, b]$ . The **upper integral** of  $f$  is

$$U(f) = \inf\{U(f, P) \mid P \in \mathcal{P}\}.$$

Similarly, the **lower integral** of  $f$  is

$$L(f) = \sup\{L(f, P) \mid P \in \mathcal{P}\}.$$

**Lemma 56.** For any bounded function  $f$  on  $[a, b]$ , it always follows that  $U(f) \geq L(f)$ .

*Proof.* TBC in HW. □

**Definition 57. (Riemann Integrability).** A bounded function  $f$  on  $[a, b]$  is **Riemann integrable** if  $U(f) = L(f)$ . Notation:

$$\int_a^b f(x) dx = U(f) = L(f).$$

## 26 November 18/20, 2019

### 26.1 Criteria for Integrability

#### Theorem: 26.1.1

**(Integrability Criterion).** A bounded function  $f$  is integrable on  $[a, b]$  if and only if, for every  $\epsilon > 0$ , there exists a partition  $P_\epsilon$  of  $[a, b]$  such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

#### Theorem: 26.1.2

If  $f$  is continuous on  $[a, b]$ , then it is integrable.

### 26.2 Integrating Functions with Discontinuities

#### Theorem: 26.2.1

If  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, and  $f$  is integrable on  $[c, b]$  for all  $c \in (a, b)$ , then  $f$  is integrable on  $[a, b]$ .

## 27 November 22, 2019

### 27.1 Properties of the Integral

#### Theorem: 29.1.1

Assume  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, and let  $c \in (a, b)$ . Then  $f$  is integrable on  $[a, b]$  iff  $f$  is integrable on  $[a, c]$  and on  $[c, b]$ . In this case,

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

#### Theorem: 29.1.2

**(Properties of the Integral).** Assume  $f, g$  are integrable on  $[a, b]$ .

1.  $f + g$  is integrable on  $[a, b]$  with  $\int_a^b (f + g) = \int_a^b f + \int_a^b g$ .

2. For  $k \in \mathbb{R}$ ,  $kf$  is integrable with  $\int_a^b kf = k \int_a^b f$ .

3. If  $m \leq f \leq M$  on  $[a, b]$ , then  $m(b - a) \leq \int_a^b f \leq M(b - a)$ .

4. If  $f \leq g$  on  $[a, b]$ , then  $\int_a^b f \leq \int_a^b g$ .

5.  $|f|$  is integrable and  $\left| \int_a^b f \right| \leq \int_a^b |f|$ .

## 27.2 The Fundamental Theorem of Calculus

### Theorem: 29.2.1

#### (Fundamental Theorem of Calculus).

1. If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable, and  $F : [a, b] \rightarrow \mathbb{R}$  satisfies  $F'(x) = f(x)$  for all  $x \in [a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

2. Let  $g : [a, b] \rightarrow \mathbb{R}$  be integrable, and for  $x \in [a, b]$  define

$$G(x) = \int_a^x g(t) dt.$$

Then  $G$  is continuous on  $[a, b]$ . Moreover, if  $g$  is continuous at  $c \in [a, b]$ , then  $G$  is differentiable at  $c$  and  $G'(c) = g(c)$ .

*Proof.* (i) Let  $P$  be a partition of  $[a, b]$  and apply the MVT to  $F$  on a typical subinterval  $[x_{k-1}, x_k]$  of  $P$ . This yields a point  $t_k \in (x_{k-1}, x_k)$  such that

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)(x_k - x_{k-1}).$$

Because  $m_k \leq f(t_k) \leq M_k$ , it follows that

$$L(f, P) \leq \sum_{k=1}^n [F(x_k) - F(x_{k-1})] \leq U(f, P).$$

Since

$$\sum_{k=1}^n [F(x_k) - F(x_{k-1})] = F(b) - F(a),$$

which is independent of  $P$ , we have

$$L(f) \leq F(b) - F(a) \leq U(f).$$

Because  $L(f) = U(f) = \int_a^b f(x) dx$ , it follows that  $\int_a^b f(x) dx = F(b) - F(a)$ .

- (ii) Take  $x > y$  in  $[a, b]$ . Then

$$|G(x) - G(y)| = \left| \int_a^x g - \int_a^y g \right| = \left| \int_y^x g \right| \leq \int_y^x |g| \leq M(x - y),$$

where  $M > 0$  is a bound for  $|g|$  on  $[a, b]$ . This shows that  $G$  is continuous.

Now, if  $g$  is continuous at  $c \in [a, b]$ , then

$$G'(c) = \lim_{x \rightarrow c} \frac{G(x) - G(c)}{x - c} = \lim_{x \rightarrow c} \frac{1}{x - c} \left( \int_c^x g(t) dt \right).$$

Given  $\epsilon > 0$ , choose  $\delta > 0$  such that  $|t - c| < \delta$  implies  $|g(t) - g(c)| < \epsilon$ . Note that  $|t - c| \leq |x - c|$  for  $t$  between  $c$  and  $x$ , and write

$$g(c) = \frac{1}{x - c} \int_c^x g(c) dt.$$

Then, for  $0 < |x - c| < \delta$ ,

$$\left| \frac{1}{x - c} \int_c^x g(t) dt - g(c) \right| = \left| \frac{1}{x - c} \int_c^x (g(t) - g(c)) dt \right| \leq \frac{1}{|x - c|} \int_c^x \epsilon dt = \epsilon.$$

Hence  $G'(c) = g(c)$ . □

## 28 November 25/27, 2019

### 28.1 Pointwise Convergence

**Definition 58.** For each  $n \in \mathbb{N}$ , let  $f_n$  be a function defined on a set  $A \subseteq \mathbb{R}$ . The sequence of functions  $(f_n)$  **converges pointwise** on  $A$  to a function  $f$  if, for all  $x \in A$ , the sequence of real numbers  $(f_n(x))$  converges to  $f(x)$ .

#### Example: 30.1.1

Let

$$f_n(x) = x^n,$$

on the set  $[0, 1]$  with  $n \in \mathbb{N}$ . Consider what happens as  $n \rightarrow \infty$ .

*Solution.* Fix  $x \in [0, 1)$ . Then  $\lim_{n \rightarrow \infty} f_n(x) = 0$ . On the other hand, if  $x = 1$ , then  $f_n(1) = 1$  for all  $n$ . It follows that  $f_n \rightarrow f$  pointwise on  $[0, 1]$ , where

$$f(x) = \begin{cases} 0, & x \in [0, 1), \\ 1, & x = 1. \end{cases}$$

■

### 28.2 Uniform Convergence

**Definition 59.** Let  $(f_n)$  be a sequence of functions defined on a set  $A$ . Then  $(f_n)$  **converges uniformly** on  $A$  to a limit function  $f$  defined on  $A$  if, for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $|f_n(x) - f(x)| < \epsilon$  whenever  $n \geq N$  and  $x \in A$ .

#### Example: 30.2.1

Let

$$g_n(x) = \frac{1}{n(1+x^2)}.$$

For any fixed  $x \in \mathbb{R}$ , we can see that  $g_n(x) \rightarrow g(x) = 0$  is the pointwise limit of the sequence  $(g_n)$  on  $\mathbb{R}$ . Is this convergence uniform?

*Solution.* Since  $0 < \frac{1}{1+x^2} \leq 1$  for all  $x \in \mathbb{R}$ ,

$$|g_n(x) - g(x)| = \left| \frac{1}{n(1+x^2)} - 0 \right| \leq \frac{1}{n}.$$

Thus, given  $\epsilon > 0$ , choose  $N > 1/\epsilon$ . Then for all  $n \geq N$  and all  $x \in \mathbb{R}$ ,

$$|g_n(x) - g(x)| < \epsilon.$$

Hence  $g_n \rightarrow 0$  uniformly on  $\mathbb{R}$ . ■

## 29 November 29, 2019

### 29.1 Cauchy Criterion

#### Theorem: 31.1.1

**(Cauchy Criterion for Uniform Convergence).** A sequence of functions  $(f_n)$  on a set  $A$  converges uniformly on  $A$  iff for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $|f_n(x) - f_m(x)| < \epsilon$  for all  $n, m \geq N$  and all  $x \in A$ .

### 29.2 Continuity Revisited

#### Theorem: 31.2.1

**(Continuous Limit Theorem).** Let  $(f_n)$  converge uniformly on  $A$  to a function  $f$ . If each  $f_n$  is continuous at  $c \in A$ , then  $f$  is continuous at  $c$ .

*Proof.* Fix  $c \in A$  and let  $\epsilon > 0$ . Choose  $N$  such that

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}$$

for all  $x \in A$ . Because  $f_N$  is continuous, there exists  $\delta > 0$  for which

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$$

whenever  $|x - c| < \delta$ . Then

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

so  $f$  is continuous at  $c$ . □

## 30 December 2, 2019

### 30.1 Convergence and Differentiation

#### Theorem: 32.1.1

**(Differentiable Limit Theorem).** Let  $f_n \rightarrow f$  pointwise on the closed interval  $[a, b]$ , and assume that each  $f_n$  is differentiable. If  $(f'_n)$  converges uniformly on  $[a, b]$  to a function  $g$ , then  $f$  is differentiable and  $f' = g$ .

*Proof.* Fix  $c \in [a, b]$  and let  $\epsilon > 0$ . We want to show that  $f'(c)$  exists and equals  $g(c)$ . Note that

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}.$$

We will find  $\delta > 0$  so that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon$$

whenever  $0 < |x - c| < \delta$ . For all  $x \neq c$  and all  $n \in \mathbb{N}$ , the triangle inequality gives

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) \right| + |f'_n(c) - g(c)|.$$

Our *goal* is to choose  $n$  so that the first and third terms are  $< \epsilon/3$ . Pick  $N_1$  such that for all  $m \geq N_1$ ,

$$|f'_m(c) - g(c)| < \frac{\epsilon}{3}.$$

Uniform convergence of  $(f'_n)$  asserts there exists  $N_2$  such that  $m, n \geq N_2$  implies

$$|f'_m(x) - f'_n(x)| < \frac{\epsilon}{3} \quad \text{for all } x \in [a, b].$$

Let  $N = \max\{N_1, N_2\}$ . Since  $f_N$  is differentiable at  $c$ , there exists  $\delta > 0$  such that

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\epsilon}{3},$$

for all  $0 < |x - c| < \delta$ .

Fix  $x$  with  $0 < |x - c| < \delta$ , and let  $m \geq N$ . Apply the MVT to  $f_m - f_N$  on  $[c, x]$ . Then there exists  $\alpha \in (c, x)$  with

$$\frac{(f_m(x) - f_N(x)) - (f_m(c) - f_N(c))}{x - c} = f'_m(\alpha) - f'_N(\alpha).$$

Taking  $m \rightarrow \infty$  and using  $f_m \rightarrow f$  and  $|f'_m(\alpha) - f'_N(\alpha)| < \epsilon/3$ , we obtain

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| \leq \frac{\epsilon}{3}.$$

Therefore, for  $0 < |x - c| < \delta$ ,

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus  $f'(c)$  exists and equals  $g(c)$ .

□