

# Cozy Home

Lily Zhang

Macalester College

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# 1 Abstract and Introduction

## 1.1 Introduction

As everyone wants to stay in a cozy home without spending much money, the development of programmable thermostats was an initial effort to reduce energy costs. Previous generations of smart home systems allow users to adjust the heating system or air conditioner remotely to save energy while away, or to learn users' behavior to raise or lower the room temperature so the house is warm or cool upon arrival.

Our mission is to develop the next generation of smart home climate control that will automatically and appropriately adjust the indoor temperature (and related variables) in response to departure and in anticipation of arrival, even when users' schedules are irregular. The algorithm we develop aims to achieve this goal and incorporates ambient outside conditions as well as relevant geographical/regional factors.

## 1.2 Problem Restatement

We consider a future generation of smart home climate control systems that will automatically and appropriately adjust the temperature of a house in response to a user's departure and in anticipation of their arrival, no matter how irregular the schedule. A smart climate system should incorporate ambient outside variations and regional conditions such as humidity, allergens, and air pollution levels.

We also require that the system integrate user preferences for temperature and other factors, such as humidity and air purity levels. Finally, we consider system requirements when more than one person is living in the home or apartment.

## 1.3 Problem Analysis

In constructing our model, we assume all input variables are continuous, so that our cost functions are at least Riemann integrable, implying that they can be evaluated numerically in the "Testing the Model" section.

The problem also requires a demonstration of an "intelligent" system, so our model must support automation. The automation must handle irregular schedules; thus our model and functions should not rely on fixed weekdays, etc. The physical realization of our model should also account for local conditions. Different filters, such as allergen filtration and draft prevention, should be considered to make the product more desirable. The problem requires careful analysis of data, which motivates the use of matrix methods.

In summary, the problem demands comfort, convenience, security, and aesthetics. The most fundamental control subsystem is the temperature control system.

A subtle difficulty is that as the system approaches a stable state, a purely time-driven mode can waste energy and overload the communication channel. Thus, we introduce an event-triggered, time-aware approach (an event-time dual-sensitive system), meaning that time conditions alone do not trigger changes; actions occur when relevant events coincide with time conditions. This improves use of the communication channel and reduces energy loss.

A flow chart for the physical construction can be described as follows:

A wireless controller (administrator) consists of a wireless signaling module, control logic, and an event detector.

The data collector (detector) measures temperature, humidity, and other variables. Through the communication channel, the data is passed to the controller to determine whether the event-time conditions are sufficient to trigger an action. If an action is triggered, we run our algorithm and send commands via the wireless controller. The controller outputs commands and actuates the devices.

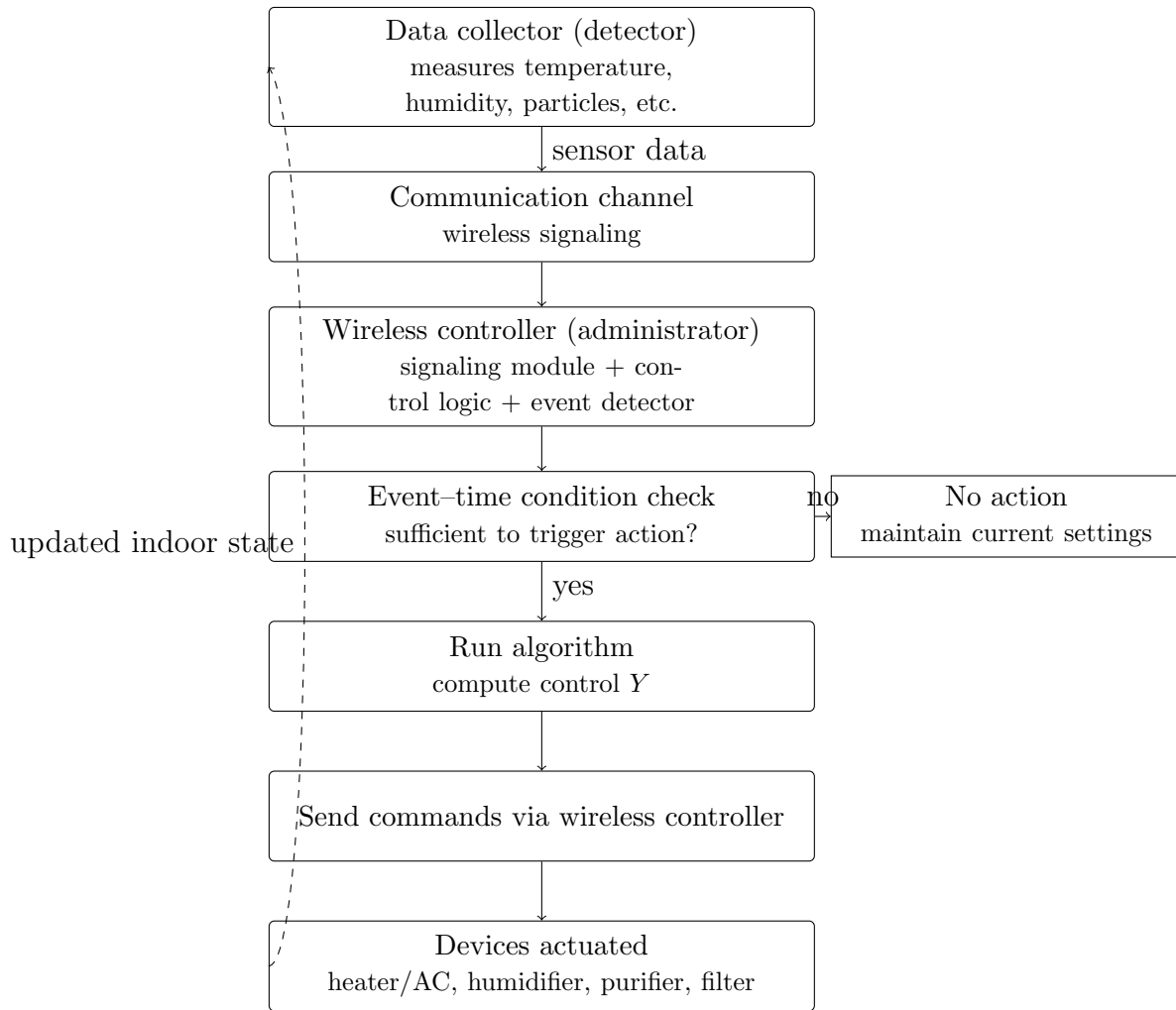


Figure 1: Flow chart of the physical system and control loop.

## 2 Assumptions and Variables

### 2.1 Assumptions and Justifications

**Assumption:** All user inputs are validated and within allowable ranges.

**Justification:** Basic validation and bounds checking prevent invalid data from corrupting learning or causing discomfort.

**Assumption:** The smart home system is connected to sensors that measure temperature, humidity, and air-pollution levels (and can access ambient/outdoor conditions).

**Justification:** Without sensor data on indoor and outdoor conditions, the controller cannot estimate the state or act adaptively.

**Assumption:** Users exhibit partially regular patterns (daily/weekly/seasonal), though any given day may be irregular; manual overrides are allowed.

**Justification:** This enables the model to learn from time features while remaining robust to irregular schedules; manual changes are treated as feedback rather than requiring reprogramming each thermostat.

**Assumption:** The discomfort of each person is locally quadratic in the relevant air-quality variables (e.g., temperature, humidity, pollutants).

**Justification:** A quadratic model simplifies the mathematics and captures local convexity around preferred conditions.

**Assumption:** Air within a room is well mixed over the controller’s sampling interval, so spatial gradients are neglected.

**Justification:** This standard lumped-parameter approximation simplifies the model; a multi-zone model could be used if needed.

**Assumption:** Outdoor weather/climate cannot be controlled.

**Justification:** The external environment is far larger than any residential system and is effectively exogenous.

**Assumption:** Over the operating range, the effects of appliances on the room climate are approximately additive (superposition).

**Justification:** We linearize around typical operating points; cross-couplings are represented via calibrated coefficients.

**Assumption:** When people make thermostat adjustments, they primarily act for comfort; energy costs are handled explicitly by the controller (e.g., via tariffs or a cost weight).

**Justification:** This avoids trying to infer energy pricing from behavior alone and yields a clearer, more identifiable model.

## 2.2 Inputs

Collected data play a very important role in our model. We assume that devices in a given smart home measure certain values and pass them to our model. Thus we have chosen common household sensors and assume the data they provide can be used in our model.

### Ambient and Room Variables

As we will see, many variables take different values indoors versus outdoors. One such variable is temperature. On a cold winter day, the indoor temperature is expected to be higher than outdoors; otherwise people inside will feel cold. We therefore define the following concepts.

**Definition 2.2.1.** An **ambient** variable applies to the conditions just outside the building. Ambient variables are out of the control of both the smart home system and its users. Ambient variables depend on the weather, which may be somewhat predictable long-term but can vary chaotically over short periods. Ambient variables are denoted with a subscript  $a$ .

**Definition 2.2.2.** A **room** variable applies to the average conditions within the room under consideration. Room variables can be affected by ambient variables, the climate control system, and the presence of people in the room.<sup>1</sup> If the room temperature is high, possible reasons include hot ambient conditions, many occupants, or the heater being on. People inside the room have preferences for certain room variables; the goal of the smart climate control system is to control these variables cost-effectively so occupants need not do so themselves. Room variables are denoted with a subscript  $r$ .

**Definition 2.2.3.** A variable is **twin** if its average value within the room is not necessarily equal to its value outside. Each twin variable  $X$  consists of two variables  $X_r$  and  $X_a$ , representing its associated room and ambient values, respectively. Variables such as temperature and humidity are twin, while others such as time and cost of electricity are not.

### Temperature

We will use the thermometer’s output—the temperature—as a main variable of discussion and a main input value for our model. When people adjust a thermostat, they most often change temperature rather than, say, humidity or air quality, so the algorithm focuses on maintaining a suitable indoor temperature. Temperature is a twin variable, consisting of ambient temperature and room temperature.

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<sup>1</sup>More specifically, the time derivatives of room variables can be affected by other variables.

**Definition 2.2.4.** The **ambient temperature**  $T_a \in \mathbb{R}$  is the outdoor temperature, in  $^{\circ}\text{C}$ , just outside the home. It varies with time and may roughly follow daily and annual cycles with random deviations. Being an ambient variable,  $T_a$  cannot be controlled by the system or its users. The date and time are assumed given.

**Definition 2.2.5.** The **room temperature**  $T_r \in \mathbb{R}$  is the average temperature, in  $^{\circ}\text{C}$ , within the room. It varies with time and is affected by variables such as  $T_a$ , internal gains, and device operation. A heater or air conditioner can alter  $T_r$ , and occupants release heat that raises  $T_r$  when present.

## Humidity

Humidity is commonly measured indoors and outdoors. It affects health and comfort and relates to allergens such as dust and pollen. Like temperature, humidity is a twin variable.

**Definition 2.2.6.** Rather than using *relative humidity* (a percentage), we use a moisture variable that is more convenient for modeling. In this document we denote it by the **water vapour pressure**  $p$  with  $p_a, p_r \in \mathbb{R}$  for ambient and room values, respectively. Relative humidity is strongly temperature-dependent because the saturation vapour pressure varies with temperature; modeling  $p$  (or, equivalently, a temperature-decoupled quantity such as specific humidity or dew point) avoids the bounded, highly non-linear dependence inherent in relative humidity. If sensors report relative humidity, a preprocessing step converts  $(\text{RH}, T)$  to  $p$  (or to an equivalent absolute measure) before use.

## Allergen Density

The air contains many minuscule particles, some of which cause allergic reactions (e.g., pollen, dust, pet hair, mold). Although the density of allergen particles is a less common metric, we assume devices can measure their density and type in a smart home. Allergen density is a twin variable.

**Definition 2.2.7.** The **allergen density**  $N$  is the number density of allergen particles in the air.  $N_a$  and  $N_r$  denote the ambient and room allergen densities, respectively. More specifically,  $N_{ai} \in \mathbb{R}$  is the number of particles of allergen type  $i$  per cubic meter of ambient air, where  $i = 1, \dots, n_{\text{allergens}}$ . The same notation applies to  $N_{ri}$  for room air.

## Air Pollution Levels

Air pollution levels can be characterized by particulate matter ( $PM_{2.5}$ ,  $PM_{10}$ ), ozone ( $O_3$ ), nitrogen dioxide ( $NO_2$ ), sulfur dioxide ( $SO_2$ ), and carbon monoxide ( $CO$ ).

**Definition 2.2.8.** The **air quality index**  $AQI \in \mathbb{R}_{\geq 0}$  denotes a scalar air-quality measure used by the model (for example,  $PM_{2.5}$  concentration in  $\mu\text{g}/\text{m}^3$  or an AQI-style rescaling). Larger values indicate worse air quality. In implementation we recommend using a directly measured concentration (e.g.,  $PM_{2.5}$ ) and, if desired, mapping it to an index for presentation.

## Climate

There are multiple variables describing the air. For compactness we group them into column vectors representing the ambient and room air states.

**Definition 2.2.9.** The ambient and room **climates**  $c_a \in \mathbb{R}^{n_{\text{input}}}$  and  $c_r \in \mathbb{R}^{n_{\text{input}}}$  are column vectors such that

$$c_a = \begin{bmatrix} T_a \\ p_a \\ AQI_a \\ N_{a1} \\ N_{a2} \\ \vdots \\ N_{an} \end{bmatrix}, \quad c_r = \begin{bmatrix} T_r \\ p_r \\ AQI_r \\ N_{r1} \\ N_{r2} \\ \vdots \\ N_{rn} \end{bmatrix},$$

where  $n_{\text{input}} = n_{\text{allergens}} + 3$ . All ambient and room air variables are summarized in these vectors. The ambient climate cannot be controlled, while the smart home climate control system acts to control the room climate.

## Floor Space

The size of the room affects the internal climate: the air volume determines how much energy is required to heat or cool it, and the surface area of the enclosure affects exchange with the environment.

**Definition 2.2.10.** The **floor space** of a room is the area of the room's floor. This is commonly reported in  $\text{ft}^2$  (in the US); for this model we use  $\text{m}^2$ . We assume the area may be entered in  $\text{ft}^2$  and converted by the algorithm before use. We also assume a room height of 3 m, allowing computation of the room volume  $V_r \in \mathbb{R}$ .

## 2.3 Outputs

After measuring the current state of the air in the room, the algorithm must decide which appliances to turn on. If the algorithm cannot control the climate through appliances, then it does not serve its purpose as a smart device. Thus we define the appliances the smart climate control system can use, as well as their effects on the room climate.

### Allergen filter

We start with the allergen filter because it is simple and illustrates how an appliance affects a state. We assume the filter draws air with a *volumetric flow*  $Q_{\text{filter}}$  ( $\text{m}^3/\text{s}$ ). Let  $\eta_i \in [0, 1]$  be the filtration efficiency for allergen type  $i$ . Under the well-mixed assumption and uniform distribution of allergens in the room, the filter removes particles at the first-order rate

$$\frac{dN_{ri}}{dt} = -\frac{\eta_i Q_{\text{filter}}}{V_r} N_{ri}, \quad 1 \leq i \leq n_{\text{allergens}}.$$

The removal rate increases with both efficiency and flow and scales inversely with room volume.

### Heater and air conditioner

We assume the space has a heater and an air conditioner. Let  $P_{\text{heater}}$  be the *heat rate* delivered to the air (W) and  $P_{A/C}$  the *heat removal rate* from the air (W). Using the volumetric heat capacity of air  $c_{P,V}$  (J/(m<sup>3</sup> K)), their direct contributions to room temperature are

$$\frac{dT_r}{dt} = \frac{P_{\text{heater}}}{V_r c_{P,V}}, \quad \frac{dT_r}{dt} = -\frac{P_{A/C}}{V_r c_{P,V}},$$

where  $P_{A/C}$  refers to heat extracted from the room air (not electrical input power) and  $c_{P,V} = \rho c_p$  (volumetric heat capacity).

### Humidifier and dehumidifier

Let  $p_r$  denote the indoor water vapour pressure (absolute moisture measure). A humidifier adds moisture at a roughly constant effective rate; we model this with a calibrated coefficient  $P_{\text{humidifier}}$  (Pa m<sup>3</sup>/s) so that

$$\frac{dp_r}{dt} = \frac{P_{\text{humidifier}}}{V_r}.$$

Here  $P_{\text{humidifier}}$  is an effective vapour-pressure injection rate times volume so that dividing by  $V_r$  yields Pa/s. A dehumidifier removes moisture from the air stream it processes. With volumetric flow  $Q_{\text{dehumidifier}}$  and effective moisture-removal efficiency  $\eta_w \in [0, 1]$ , a first-order removal model gives

$$\frac{dp_r}{dt} = -\frac{\eta_w Q_{\text{dehumidifier}}}{V_r} p_r.$$

### Purifier

For a general pollutant scalar (e.g., *PM2.5* concentration; we denote it by *AQI* as a modelled scalar), a purifier with volumetric flow  $Q_{\text{purifier}}$  and particle removal efficiency  $\eta_p$  yields

$$\frac{dAQI}{dt} = -\frac{\eta_p Q_{\text{purifier}}}{V_r} AQI.$$

(When implementing, we recommend using a directly measured concentration such as *PM2.5*; the same form applies.)

### Overall effect on climate

Taking all outputs into account, we collect device effects in matrices. Let  $Y \in \mathbb{R}^{n_{\text{output}}}$  indicate on/off states (1/0), where  $n_{\text{output}}$  is the number of device channels. Using two sparse matrices  $M_1$  (additive effects) and  $M_2$  (multiplicative first-order removal), and the Hadamard product  $\circ$ , an example with  $n_{\text{allergens}} = 2$  is

$$\frac{dc_r}{dt} = \begin{bmatrix} \frac{P_{\text{heater}}}{c_{P,V}} & -\frac{P_{A/C}}{c_{P,V}} & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{\text{humidifier}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_{\text{heater}} \\ Y_{\text{AC}} \\ Y_{\text{humidifier}} \\ Y_{\text{dehumidifier}} \\ Y_{\text{purifier}} \\ Y_{\text{filter}} \end{bmatrix} \Big/ V_r$$

$$+ c_r \circ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_{\text{dehumidifier}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_{\text{purifier}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -P_{\text{filter},1} \\ 0 & 0 & 0 & 0 & 0 & -P_{\text{filter},2} \end{bmatrix} \begin{bmatrix} Y_{\text{heater}} \\ Y_{\text{AC}} \\ Y_{\text{humidifier}} \\ Y_{\text{dehumidifier}} \\ Y_{\text{purifier}} \\ Y_{\text{filter}} \end{bmatrix} / V_r,$$

where the five rows correspond to, in order: temperature  $T_r$ , water vapour pressure  $p_r$ , pollutant scalar ( $AQI/PM2.5$ ), allergen 1 density, and allergen 2 density. Here

$$P_{\text{dehumidifier}} := \eta_w Q_{\text{dehumidifier}}, \quad P_{\text{purifier}} := \eta_p Q_{\text{purifier}}, \quad P_{\text{filter},i} := \eta_i Q_{\text{filter}},$$

so that dividing by  $V_r$  gives first-order rate constants ( $\text{s}^{-1}$ ). Additional cross-effects (e.g., device waste heat or built-in filtration in the heater) can be represented by inserting small nonzero entries in  $M_1$  or  $M_2$  as appropriate. For compactness we write

**Definition 2.3.1.** The device-induced change in room climate is summarized by

$$\frac{dc_r}{dt} = \frac{M_1 Y + c_r \circ M_2 Y}{V_r}.$$

## 2.4 Modelling Human Variables

People prefer different indoor conditions, and a smart climate control system should adapt to the preferences of the occupants in the room.

**Definition 2.4.1.** The **preferred climate**  $c_{pi} \in \mathbb{R}^{n_{\text{input}}}$  of person  $i$  is a (possibly time-varying) column vector representing person  $i$ 's preferred indoor state.

Occupants also differ in how sensitive they are to deviations from their preferred conditions.

**Definition 2.4.2.** The **sensitivity**  $S_i \in \mathbb{R}^{n_{\text{input}} \times n_{\text{input}}}$  of person  $i$  is a symmetric positive definite (SPD) matrix that may depend on time. It defines a quadratic **discomfort**

$$D_i = (c_r - c_{pi})^T S_i (c_r - c_{pi}) \geq 0,$$

which is strictly convex in  $c_r$  because  $S_i \succ 0$ .

A single scalar weight would not capture different sensitivities to different factors (e.g., temperature vs. moisture vs. particles). A diagonal  $S_i$  captures per-variable weights but not cross-sensitivities (e.g., interactions between temperature and moisture); allowing a full symmetric  $S_i$  permits such interactions.

**Definition 2.4.3.** The **cost function** at time  $t$  is

$$C(t) = \sum_{i=1}^{n_p} D_i(c_r(t)) + C_{\text{energy}}(Y(t)),$$

the sum of discomforts plus the cost of running devices. Here  $n_p$  is the number of occupants. When deriving the comfort optimum below, we hold  $C_{\text{energy}}$  fixed and minimize the discomfort term  $\sum_i D_i$ .

**Lemma 2.4.4.** The climate that minimizes total discomfort,

$$c_{\text{opt}} = \arg \min_c \sum_{i=1}^{n_p} (c - c_{pi})^T S_i (c - c_{pi}),$$

is

$$c_{\text{opt}} = \left( \sum_{i=1}^{n_p} S_i \right)^{-1} \left( \sum_{i=1}^{n_p} S_i c_{pi} \right).$$

*Proof.* The objective is strictly convex since each  $S_i \succ 0$ . Its gradient is  $\nabla_c \sum_i (c - c_{pi})^T S_i (c - c_{pi}) = 2 \sum_i S_i (c - c_{pi})$ . Setting the gradient to zero gives  $\sum_i S_i c = \sum_i S_i c_{pi}$ , hence the stated solution.  $\square$

**Lemma 2.4.5.** Let  $S_{\text{eff}} := \sum_{i=1}^{n_p} S_i$  and let  $c_{\text{opt}}$  be as in Lemma 2.4.4. Then the total discomfort can be written as

$$\sum_{i=1}^{n_p} D_i = (c_r - c_{\text{opt}})^T S_{\text{eff}} (c_r - c_{\text{opt}}) + \sum_{i=1}^{n_p} (c_{\text{opt}} - c_{pi})^T S_i (c_{\text{opt}} - c_{pi}).$$

*Proof.* Expand each quadratic and complete the square around  $c_{\text{opt}}$ ; cross terms cancel by the optimality condition  $\sum_i S_i (c_{\text{opt}} - c_{pi}) = 0$ .  $\square$

**Definition 2.4.6.** The **effective sensitivity** is

$$S_{\text{eff}} = \sum_{i=1}^{n_p} S_i \in \mathbb{R}^{n_{\text{input}} \times n_{\text{input}}},$$

and the **minimum aggregate discomfort** (a constant with respect to  $c_r$ ) is

$$D_{\text{min}} = \sum_{i=1}^{n_p} (c_{\text{opt}} - c_{pi})^T S_i (c_{\text{opt}} - c_{pi}).$$

Thus,

$$\sum_{i=1}^{n_p} D_i = (c_r - c_{\text{opt}})^T S_{\text{eff}} (c_r - c_{\text{opt}}) + D_{\text{min}}.$$

The constant  $D_{\text{min}}$  does not affect control decisions; the controllable part depends only on  $c_r - c_{\text{opt}}$  and  $S_{\text{eff}}$ .

## 2.5 Modelling the Room

In order to suit the needs of the user, the system must be able to predict the characteristics of the room so it knows approximately how to control its climate. Many external, uncontrollable factors (such as allergens carried by wind) may affect the room's climate alongside the system's actions, and these effects must be modelled for effective control.

## Heat conduction through walls

Heat conducts through the building envelope between the room and the external environment. A general description uses the heat equation,

$$\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot (k \nabla T) = \dot{q}_V,$$

but here we use a standard lumped (well-mixed) first-order model based on Fourier conduction through a wall of thickness  $\epsilon$  and area  $A$  with conductivity  $k$ . The conductive heat flow (taken positive from room to outside) is

$$P_{\text{cond}} = \frac{kA}{\epsilon} (T_r - T_a) = U (T_r - T_a),$$

where  $U := kA/\epsilon$  (W/K). Let

$$C_{\text{eff}} := V_r c_{P,V} + C_{\text{envelope}}$$

be the *effective* thermal capacitance (J/K) of the room air plus envelope/furnishings (with  $c_{P,V} = \rho c_p$ ). Energy balance gives

$$C_{\text{eff}} \frac{dT_r}{dt} = -U (T_r - T_a), \quad \text{i.e.} \quad \frac{dT_r}{dt} = -\alpha (T_r - T_a), \quad \alpha := \frac{U}{C_{\text{eff}}} > 0.$$

This ordinary differential equation has an exponential solution approaching the asymptote  $T_a$ . Adding windows, doors, or other conductive paths changes  $U$  (and thus  $\alpha$ ).

## Air convection through open windows

When windows are open, indoor air exchanges with outdoor air at a rate that depends on wind and openings. Under the well-mixed assumption, any scalar component of the indoor climate is drawn toward the ambient value at a rate proportional to the volumetric flow  $Q_{\text{window}}$ :

$$\frac{dc_r}{dt} = -\frac{Q_{\text{window}}}{V_r} (c_r - c_a).$$

There is always some infiltration/exfiltration even with windows closed. For algorithmic purposes we model two regimes: a small baseline exchange when closed and a much larger exchange when open; typical open-window mixing rates satisfy  $Q_{\text{window}}/V_r \sim 10^{-2} \text{ s}^{-1}$  (order-of-minutes time constant), causing  $c_r$  to approach  $c_a$  rapidly and largely outside the control authority of the system.

## Climate alteration by humans

Human metabolism produces heat that increases  $T_r$  and also introduces moisture and particles. Let  $P_p$  (W) be the total sensible heat from occupants. The temperature balance becomes

$$C_{\text{eff}} \frac{dT_r}{dt} = -U (T_r - T_a) + P_p, \quad \text{so} \quad \frac{dT_r}{dt} = -\alpha \left( T_r - \left( T_a + \frac{P_p}{U} \right) \right).$$

Thus occupancy shifts the steady-state temperature by  $P_p/U$ . Analogously, occupant sources for other climate components (e.g., moisture, particles) act like offsets in their respective balances.

Collecting conduction, mixing, and human effects for the vector climate, we write

$$\frac{dc_r}{dt} = -\sigma \left( c_r - (c_a + \Delta c_h) \right),$$

where  $\sigma$  is a diagonal matrix of first-order rates (e.g.,  $\alpha$  for temperature, air-exchange rates for other components), and  $\Delta c_h$  is a vector capturing effective occupant source offsets. In the two-regime window model,  $\sigma$  switches between a small “closed” value and a larger “open” value. Assuming  $\sigma$  and  $\Delta c_h$  are (piecewise) constant, differentiating yields

$$\frac{d^2 c_r}{dt^2} = -\sigma \left( \frac{dc_r}{dt} - \frac{dc_a}{dt} \right),$$

which can be used to estimate  $\sigma$  from time series of  $c_r$  and  $c_a$ ; in practice we low-pass both signals and include a small ridge  $\varepsilon > 0$  in the denominator for numerical stability.

## 2.6 Training the model

The principal time-varying quantity we learn from manual adjustments is the optimal climate  $c_{\text{opt}}(t)$ . We treat the effective sensitivity  $S_{\text{eff}}$  as a symmetric positive definite matrix chosen from defaults (or adjusted via explicit user feedback), and we handle energy costs explicitly elsewhere in the objective rather than attempting to infer them from behavior.

To train the algorithm, we use data from previous manual adjustments of the thermostat/devices. Each adjustment  $i$  is described by: the time  $t_i$  of the adjustment, the indoor climate  $c_{ri}$  just before the adjustment, and the new output  $Y_i$  that was set (on/off vector). From Definition 2.3.1, the instantaneous rate of change produced by that output is

$$\dot{c}_{ri} = \left. \frac{dc_r}{dt} \right|_{(c_{ri}, Y_i)} = \frac{M_1 Y_i + c_{ri} \circ M_2 Y_i}{V_r}.$$

We convert each manual action into a *pseudo-target preference* by projecting a short time  $\tau$  (e.g., several minutes) forward along this direction:

$$\tilde{c}_{pi} := c_{ri} + \tau \dot{c}_{ri}.$$

(If explicit setpoints are available for any coordinate—e.g., a temperature setpoint—use those setpoint values to overwrite the corresponding entries of  $\tilde{c}_{pi}$ .)

### Time weighting

Because preferences often depend on time-of-day, day-of-week, and season, we weight past adjustments by their temporal similarity to the current time  $t$ . We use periodic Gaussian kernels and *multiply* them across periods:

$$W_{ii}(t) = \prod_{T \in \{\text{day, week, year}\}} \exp \left( -a_T \sin^2 \left( \pi \frac{t - t_i}{T} \right) \right),$$

where  $a_T > 0$  controls the width for each period (larger  $a_T$  gives narrower peaks). Take  $t$  in days so  $T \in \{1, 7, 365\}$  (or use consistent seconds). For robustness, compute in log-space or cap  $\sum_T a_T$  so weights do not underflow; ensure the denominator below has a small floor if needed.

### Time-weighted least squares (local model)

Let  $\tilde{C} = [\tilde{c}_{p1} \ \tilde{c}_{p2} \ \cdots \ \tilde{c}_{pn_{\text{adjustments}}}] \in \mathbb{R}^{n_{\text{input}} \times n_{\text{adjustments}}}$  collect the pseudo-targets as columns and let  $W(t)$  be the diagonal weight matrix with entries  $W_{ii}(t)$ . For a local constant model (intercept only), the time-varying optimal climate estimate is the weighted average

$$c_{\text{opt}}(t) = \frac{\tilde{C} W(t) \mathbf{1}}{\mathbf{1}^T W(t) \mathbf{1}},$$

where  $\mathbf{1}$  is an all-ones vector of length  $n_{\text{adjustments}}$ . This provides a fast, closed-form update suitable for online use.

*Optional extension (feature regression).* If desired, include features such as  $\sin(2\pi t/T)$  and  $\cos(2\pi t/T)$  for  $T \in \{\text{day, week, year}\}$  (and ambient variables) in a design matrix  $\Phi \in \mathbb{R}^{n_{\text{adjustments}} \times m}$ . Then a ridge-regularized weighted least squares fit yields parameters  $\Theta \in \mathbb{R}^{m \times n_{\text{input}}}$ :

$$\Theta = (\Phi^T W(t) \Phi + \lambda I)^{-1} \Phi^T W(t) \tilde{C}^T, \quad \text{and} \quad c_{\text{opt}}(t) = \Theta^T \phi(t),$$

with feature vector  $\phi(t)$  at the current time and regularization  $\lambda \geq 0$ .

*Notes.* (i) Estimating  $S_{\text{eff}}$  from device toggles alone is not identifiable (comfort and energy trade-offs confound), so we keep  $S_{\text{eff}}$  fixed SPD (possibly diagonal) or adjust it via explicit feedback.

(ii) Temporal weights are normalized implicitly by the denominator; ensure enough coverage across the day/week/year so the denominator is non-negligible (apply a floor if needed).

(iii) Choose  $\tau$  small enough that the linearized device effect is a good local proxy for the user's intended direction.

## 2.7 Using the model

Now that the algorithm can estimate the time-varying optimal climate  $c_{\text{opt}}(t)$  (and we have a fixed SPD  $S_{\text{eff}}$ ), it must act to minimize cost. First, estimate the diagonal rate matrix  $\sigma$  (conduction/mixing) as in the previous subsection. If *any* diagonal entry of  $\sigma$  exceeds a threshold  $\sigma_{\text{max}} \approx 10^{-3} \text{ s}^{-1}$  (or per-channel thresholds), the windows are likely open and active climate control has little authority; set the output  $Y$  to all zeros (turn devices off). Otherwise, proceed.

Define the natural (unforced) room drift

$$g(c_r) := -\sigma \left( c_r - (c_a + \Delta c_h) \right),$$

and the device-induced change (from Definition 2.3.1)

$$h(c_r, Y) := \frac{M_1 Y + c_r \circ M_2 Y}{V_r}.$$

The combined instantaneous rate is  $\dot{c}_r = g(c_r) + h(c_r, Y)$ .

To include energy explicitly, define the estimated energy rate

$$E(Y) := \sum_x P_x^{(\text{elec})} Y_x,$$

where  $P_x^{(\text{elec})}$  are device electrical power rates (or equivalent tariff-weighted costs). Let  $\beta_E > 0$  be the comfort–energy trade-off weight.

**Instantaneous test.** Use the *instantaneous* cost derivative

$$\frac{dJ}{dt} = \underbrace{2(c_r - c_{\text{opt}})^T S_{\text{eff}} [g(c_r) + h(c_r, Y)]}_{dD/dt} + \beta_E E(Y).$$

If  $dJ/dt < -\theta_J$  (tunable threshold; note  $dJ/dt$  units are cost per second), apply  $Y$ ; otherwise keep  $Y = \mathbf{0}$ .

**Short-horizon test.** Alternatively, for a small step  $\Delta t$ ,

$$c_r^+ = c_r + \Delta t [g(c_r) + h(c_r, Y)], \quad \Delta D(Y) = (c_r^+ - c_{\text{opt}})^T S_{\text{eff}} (c_r^+ - c_{\text{opt}}) - (c_r - c_{\text{opt}})^T S_{\text{eff}} (c_r - c_{\text{opt}}),$$

and

$$J(Y) = \Delta D(Y) + \beta_E E(Y) \Delta t.$$

Select the action with the smallest  $J(Y)$ ; require  $J(Y) < -\theta$  to change state. Tune  $\theta$  (units of cost) so that manual changes typically correspond to  $|J| \approx 1$ .

Evaluate a small candidate set for  $Y$  (include the *current*  $Y$ , the do-nothing  $\mathbf{0}$ , single-device toggles, and optionally a few two-device combinations) and choose the action with the largest improvement. To avoid rapid toggling, apply hysteresis and dwell times: only change a device state if the improvement exceeds the threshold by a margin and the device has been in its current state for at least a minimum on/off time.

**Definition 2.7.1.** The **time sequence** is the set  $\{t_1, t_2, \dots\}$  of instants at which the detector/controller processes data, with nominal sampling period  $\Delta T$ . In practice, the detector may report only at some of these times (event-driven transmission), so the *real* time sequence is a subset of the nominal sequence.

## 2.8 Pseudocode

Two sets of instructions are required: one for when a manual adjustment is made and one for the automated loop. Manual events are logged so the system can learn; the automated loop uses these logs to act.

### Initial Setup

1. Initialize nominal device parameters in  $M_1$  and  $M_2$  (powers, flows, efficiencies). Set  $S_{\text{eff}} \succ 0$  (e.g., diagonal with temperature weight  $>$  humidity  $>$  particles). Choose thresholds and constants:  $\sigma_{\text{max}} \approx 10^{-3} \text{ s}^{-1}$ , comfort thresholds  $\theta, \theta_J > 0$ , projection horizon  $\tau$  (minutes), step size  $\Delta t$ , and minimum on/off dwell times.

2. Initialize an empty dataset  $\mathcal{D} = \emptyset$  for past adjustments; begin sensing/logging indoor  $c_r$  and ambient  $c_a$  time series.

### Manual/Human Adjustment

1. When the user changes settings, record the time  $t_i$ , the current indoor climate  $c_{ri}$  (just before the change), the ambient climate  $c_{ai}$ , and the new output vector  $Y_i$  (device on/off states).
2. Compute the device-induced rate at that state (Definition 2.3.1):

$$\dot{c}_{ri} = \left. \frac{dc_r}{dt} \right|_{(c_{ri}, Y_i)} = \frac{M_1 Y_i + c_{ri} \circ M_2 Y_i}{V_r}.$$

3. Form a pseudo-target preference by projecting a short time  $\tau$  forward:

$$\tilde{c}_{pi} = c_{ri} + \tau \dot{c}_{ri}.$$

If an explicit setpoint was chosen (e.g., temperature), overwrite that coordinate in  $\tilde{c}_{pi}$  with the setpoint value.

4. Append  $(t_i, \tilde{c}_{pi})$  to the dataset  $\mathcal{D}$ .

### Automated Adjustment

1. Read current indoor  $c_r$  and ambient  $c_a$ . From the last few minutes of samples, estimate  $\frac{dc_r}{dt}$ ,  $\frac{dc_a}{dt}$ , and  $\frac{d^2c_r}{dt^2}$  using a small windowed least-squares fit (to reduce noise).
2. Estimate the diagonal rate matrix (elementwise) from the well-mixed model

$$\frac{d^2c_r}{dt^2} = -\sigma \left( \frac{dc_r}{dt} - \frac{dc_a}{dt} \right), \quad \sigma \approx \max \left( 0, -\frac{d^2c_r/dt^2}{dc_r/dt - dc_a/dt + \varepsilon} \right), \quad \varepsilon > 0.$$

3. If any diagonal entry of  $\sigma$  exceeds  $\sigma_{\max}$  (likely open-window/high-exchange regime), set  $Y = \mathbf{0}$  (turn devices off) and continue to the next cycle.
4. Compute temporal weights for the current time  $t$  using periodic kernels (product across periods):

$$W_{ii}(t) = \prod_{T \in \{\text{day, week, year}\}} \exp \left( -a_T \sin^2 \left( \pi \frac{t - t_i}{T} \right) \right),$$

with user-chosen widths  $a_T > 0$  (or  $a_T = n_{\text{adjustments}}$ ).

5. Estimate the current optimal climate  $c_{\text{opt}}(t)$  as a weighted average of pseudo-targets. Stack  $\tilde{c}_{pi}$  as columns of  $\tilde{C} \in \mathbb{R}^{n_{\text{input}} \times n_{\text{adjustments}}}$  and let  $W(t)$  be diagonal with entries  $W_{ii}(t)$ :

$$c_{\text{opt}}(t) = \frac{\tilde{C} W(t) \mathbf{1}}{\mathbf{1}^T W(t) \mathbf{1}}.$$

6. Define natural drift and device effect:

$$g(c_r) := -\sigma(c_r - (c_a + \Delta c_h)), \quad h(c_r, Y) := \frac{M_1 Y + c_r \circ M_2 Y}{V_r}.$$

7. Evaluate a small candidate set of actions  $\mathcal{U}$  (include the current  $Y$ ,  $\mathbf{0}$ , each single-device-on vector, and optionally a few two-device combinations). Either use the instantaneous rule

$$\dot{J}(Y) = 2(c_r - c_{\text{opt}})^T S_{\text{eff}} [g(c_r) + h(c_r, Y)] + \beta_E E(Y),$$

and select the  $Y$  with  $\dot{J}(Y) < -\theta_J$ , or perform a short-horizon step

$$c_r^+ = c_r + \Delta t [g(c_r) + h(c_r, Y)], \quad \Delta D(Y) = (c_r^+ - c_{\text{opt}})^T S_{\text{eff}}(c_r^+ - c_{\text{opt}}) - (c_r - c_{\text{opt}})^T S_{\text{eff}}(c_r - c_{\text{opt}}),$$

with

$$J(Y) = \Delta D(Y) + \beta_E E(Y) \Delta t,$$

and select the  $Y$  with  $J(Y) < -\theta$  (otherwise choose  $Y = \mathbf{0}$ ).

8. Apply hysteresis and dwell times (only change a device state if the improvement exceeds the threshold by a margin and the device has been in its current state long enough), then actuate the chosen  $Y$ .

## 3 Strengths and Weaknesses Compared to Previous Systems

### 3.1 Strengths

Unlike basic thermostats or uncoordinated air-quality devices, our design coordinates multiple variables and actions. Key strengths include:

- **Coordinated multi-variable control:** temperature, moisture (vapour pressure), and particles/allergens are modelled together, so the controller can run only what is needed rather than operating subsystems independently.
- **Event-triggered, time-aware control:** actions are taken when conditions warrant, with temporal weighting of learned preferences and hysteresis/dwell times to limit chattering and reduce energy waste.
- **Learning of time-varying preferences:** the system estimates  $c_{\text{opt}}(t)$  from user adjustments with periodic (day/week/year) kernels and aggregates multiple occupants via  $S_{\text{eff}}$ .
- **Open-window/high-exchange detection:** estimating  $\sigma$  helps avoid futile actuation when the indoor state rapidly follows the ambient state.
- **Modular formulation:** the sparse matrices  $M_1$  and  $M_2$  make it easy to extend the model with additional devices or cross-effects.

## 3.2 Weaknesses

There are still factors the system may not be able to detect or represent fully:

- **Sensor and identification limits:** not all allergens are measurable in practice; consumer sensors may report only a few particle metrics (e.g.,  $PM_{2.5}$ ). We also treat  $S_{\text{eff}}$  as configured (not inferred) to avoid identifiability issues.
- **Well-mixed, single-zone assumption:** the model neglects spatial gradients and multi-room flows; performance can degrade in large or partitioned spaces.
- **User input burden:** users may need to declare sensitivities or allergen categories; this can be inconvenient for some (e.g., seniors) without a streamlined UI.
- **Model simplifications:** device efficiencies, cross-couplings (e.g., waste heat, embedded filtration), and window-exchange estimates are simplified and must be calibrated.

## 4 Further Work

1. Test the model and controller with a simulator (e.g., forward Euler with a stable step size, or RK2) to validate tracking, energy use, and anti-chatter behavior.
2. Collect additional user data and report effectiveness; calibrate  $M_1$ ,  $M_2$ , and temporal weighting widths using held-out periods (and short device step-tests).
3. Extend and generalize to multi-zone layouts, additional sensors (e.g.,  $CO_2$  for occupancy inference), and tariff-aware energy cost terms.

# Appendix

**Table 1: Major variables in model**

Variable	Description
$c_a$	Ambient climate vector
$c_r$	Room climate vector
$V_r$	Room volume
$P_x$	Power/flow parameter of appliance $x$
$S_{\text{eff}}$	Effective sensitivity matrix
$c_{\text{opt}}$	Time-varying optimal climate
$D$	Total discomfort
$Q_{\text{window}}$	Volumetric flow through windows
$\sigma$	Conduction/mixing rate matrix
$Y$	Output (device on/off) vector
$dc_r/dt$	Rate of change of room climate
$n_x$	Number of items of type $x$
$W$	Time-weight matrix for learning
$M_1$ and $M_2$	Additive and multiplicative device effects
$\tilde{C}$	Matrix of pseudo-target preferences from adjustments
$\mathcal{D}$	Dataset of past adjustments $(t_i, \tilde{c}_{pi})$