Baire Category Theorem and its Applications

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Abstract

We present the Baire Category Theorem (BCT) and several of its consequences in analysis. After recalling the basic notions of metric and Banach spaces, we state BCT and record equivalent formulations (e.g., that countable intersections of dense open sets are dense) together with the terminology of meager and generic sets. We give a proof of BCT and then apply it to (i) points of continuity for pointwise limits of continuous functions, (ii) the genericity of continuous nowhere differentiable functions in $C[0,1]$, and (iii) the Banach–Steinhaus (Uniform Boundedness) Theorem. Finally, we deduce the Open Mapping and Closed Graph Theorems and illustrate them with an application to Grothendieck's theorem on closed subspaces of L^p.

1 Introduction

1.1 Overview

In this paper, we study the Baire Category Theorem (BCT) and its applications in analysis.

The Baire Category Theorem in its more general form states that any complete metric space cannot be the countable union of nowhere dense subsets [ER]. Although this may seem intuitive when first encountered, the theorem is mathematically elegant and plays an important role in many proofs.

We start with relevant definitions of metric spaces and Banach spaces. We then state BCT and give equivalent formulations of BCT, and explain the categories of sets Baire defined, followed by a proof of BCT.

In the next section, we discuss the consequences of BCT and its applications in analysis and continuity. Specifically, we show that the set of continuous, nowhere differentiable functions is generic (comeager) in $C[0,1]$. Then, we prove the Banach–Steinhaus Theorem (Uniform Boundedness Principle).

Then, from Baire's Category Theorem, we deduce two further general results: the Open Mapping and Closed Graph Theorems. For each theorem, we provide an example of their use, including a proof of Grothendieck's theorem on closed subspaces of L^p.

2 Definitions

2.1 Banach Spaces.

Definition 1. *(**Normed Vector Space**) Let V denote a (possibly infinite-dimensional) vector space over a field $\mathbb{F}$ (either $\mathbb{C}$ or $\mathbb{R}$). A map $||\cdot|| : V \to [0, \infty)$ satisfies:*

1. $||v|| = 0 \iff v = 0$,
2. $||\alpha v|| = |\alpha|\,||v||$ *for all* $\alpha \in \mathbb{F}$ *and* $v \in V$,
3. $||v + w|| \leq ||v|| + ||w||$ *for all* $v, w \in V$.

We call $||\cdot||$ a norm and $(V, ||\cdot||)$ a normed vector space.

Given a norm, it is an easy verification that $d(u, v) = ||u - v||$ defines a *metric* on V.

Definition 2. *(**Banach Space**) If the normed vector space $(V, ||\cdot||)$ is complete with respect to the metric induced by the norm (i.e., each Cauchy sequence $\{v_n\}$ converges to some point in V), then we call V a Banach space.*

Definition 3. *(**Bounded**) A linear map $T : \mathcal{X} \to \mathcal{Y}$ between two normed vector spaces is called bounded if there is some $C \geq 0$ such that $||Tx|| \leq C||x||$ for every $x \in \mathcal{X}$. (Note this differs from the usual set-theoretic notion of a bounded map; a nonzero linear map cannot be bounded in that sense.) We denote the space of all bounded linear maps by $L(\mathcal{X}, \mathcal{Y})$, and define the operator norm by*

$$||T|| = \sup \left\{ \frac{||Tx||}{||x||} : x \neq 0 \right\}.$$

2.2 Examples of Banach Spaces

Example 4. *On a measure space, the L^p spaces $(1 \leq p \leq \infty)$ with their usual norms are Banach spaces.*

Example 5. *The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces.*

Example 6. *If K is a compact metric space, then $C(K)$ with the supremum norm is a Banach space, and for each $x \in K$ the evaluation operator $\pi_x(f) = f(x)$ is bounded and continuous.*

2.3 Nowhere Dense Sets.

Definition 7. *(**Open Sets**) Let X be a metric space with metric d. Let $B_r(x)$ denote the open ball centered at x and of radius r,*

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

A set $O \subset X$ is open if for every $x \in O$ there exists $r > 0$ so that $B_r(x) \subset O$. We write $\overline{B}(x,r) = \{y \in X : d(x,y) \leq r\}$ for the closed ball.

Definition 8. ***(Closed Sets)*** *A set is closed if its complement is open.*

Definition 9. ***(Interior)*** *The interior E° of a set $E \subset X$ is the union of all open sets contained in E. The closure $\bar{E}$ of E is the intersection of all closed sets containing E.*

Lemma 10. *E° is the "largest" open set contained in E and $\bar{E}$ is the "smallest" closed set containing E.*

Definition 11. ***(Nowhere dense sets)*** *A set $E \subset X$ is nowhere dense if $\text{int}(\bar{E}) = \varnothing$. Equivalently, E is not dense in any nonempty open set.*

Example 12. *Examples of nowhere dense sets.*

1. *A point in $\mathbb{R}^d$ is nowhere dense in $\mathbb{R}^d$.*

2. *The middle-third Cantor set is nowhere dense in $\mathbb{R}$.*

3 Baire Category Theorem

Definition 13. *A set $E \subset X$ is of the* ***first category*** *in X if E is a countable union of nowhere dense sets in X. A set of the first category is sometimes said to be "meager". A set E that is not of the first category in X is referred to as being of the* ***second category*** *in X.*

Definition 14. *A set $E \subset X$ is* ***generic*** *if its complement is of the first category.*

Theorem 15. ***(Baire Category Theorem)****. Let $E_1, E_2, \ldots$ be a (countable) sequence of subsets of a complete metric space X. If $\bigcup_n E_n$ contains a ball B, then at least one of the E_n is dense in a sub-ball B' of B (and in particular is not nowhere dense). In contrapositive form: the countable union of nowhere dense sets cannot contain a ball.*

Theorem 16. ***(Equivalent Formulations of BCT)*** *Let X be a complete metric space. Then:*

(a) A meager set has empty interior.

(b) The complement of a meager set is dense.

(c) A countable intersection of dense open sets is dense.

Corollary 17. ***(Weak Form of the Baire Category Theorem)*** *Let X be a nonempty complete metric space. Then:*

(a) X cannot be written as a countable union of nowhere dense sets.

(b) If X is written as a countable union of closed sets, then at least one of those closed sets has nonempty interior.

(c) A countable intersection of dense open sets is nonempty.

Lemma 18. ***(Cantor)*** *Let X be a complete metric space, and let $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ be a decreasing sequence of nonempty closed subsets of X, with $\operatorname{diam}(F_n) \to 0$. Then there exists a point $x \in X$ such that $\cap_{n=1}^{\infty} F_n = \{x\}$. In particular, $\cap_{n=1}^{\infty} F_n \neq \varnothing$.*

Proof. In each set F_n, choose a point x_n. Then the sequence (x_n) is Cauchy: for $m, n \geq N$ we have $x_m, x_n \in F_N$, so $d(x_m, x_n) \leq \operatorname{diam}(F_N) \to 0$ as $N \to \infty$. Since X is complete, (x_n) has a limit x. Moreover, $x_n \in F_N$ for all $n \geq N$ and F_N is closed, hence $x \in F_N$ for every N, so $x \in \cap_{N=1}^{\infty} F_N$. Finally,

$$\operatorname{diam}(\cap_{N=1}^{\infty} F_N) \leq \inf_{N \geq 1} \operatorname{diam}(F_N) = 0,$$

so $\cap_{n=1}^{\infty} F_n = \{x\}$. □

Corollary 19. *The Baire Category Theorem is equivalent to the claim that in a complete metric space, a countable intersection of dense open sets is dense.*

Proof. Let $\{A_n\}_{n \geq 1}$ be a sequence of open dense sets in X and set $A = \cap_{n=1}^{\infty} A_n$. To show A is dense, let $W \subset X$ be nonempty and open; we will show $W \cap A \neq \varnothing$.
Since A_1 is open and dense, $W \cap A_1$ is nonempty and open. Choose $x_1 \in W \cap A_1$ and $r_1 \in (0, 1)$ with $\overline{B(x_1, r_1)} \subset W \cap A_1$. Inductively, having chosen x_{n-1} and r_{n-1}, the set $A_n \cap B(x_{n-1}, r_{n-1})$ is nonempty and open (because A_n is dense and open), so choose $x_n \in A_n \cap B(x_{n-1}, r_{n-1})$ and $r_n \in (0, 2^{-n})$ with

$$\overline{B(x_n, r_n)} \subset A_n \cap B(x_{n-1}, r_{n-1}).$$

Then the closed balls are nested and diam $\overline{B(x_n, r_n)} \leq 2r_n \to 0$, so by Cantor's lemma their intersection is nonempty. Any x in this intersection lies in W and in every A_n, hence $x \in W \cap A$. Since W was arbitrary, A is dense. □

Corollary 20. *Any nonempty complete metric space without isolated points is uncountable. (In particular, this shows that the Baire Category Theorem need not hold for incomplete metric spaces such as the rationals $\mathbb{Q}$.)*

Proof. Suppose X is countable. Every singleton $\{x\}$ is closed with empty interior (since x is not isolated), so each singleton is nowhere dense in X. Then $X = \cup_{x \in X} \{x\}$ is meager in itself, contradicting Baire's theorem. □

Definition 21. *We define the* ***oscillation*** *of a function f at a point x by*

$$\operatorname{osc}(f)(x) = \lim_{r\downarrow 0} \omega(f)(r,x), \quad \textit{where} \ \ \omega(f)(r,x) = \sup_{y,z\in B_r(x)} |f(y)-f(z)|.$$

This notion of category describes "smallness" in purely topological terms. It reflects the idea that elements of a set of the first category are to be thought of as "exceptions," while those of a generic set are to be considered "typical." In particular, we will exhibit a generic (comeager) set of measure zero; likewise, there exist first-category sets in $[0,1]$ with full measure.

Example 22. *Let $\{x_j\}_{j=1}^{\infty}$ denote an enumeration of the rational numbers in $\mathbb{R}$, and consider the sets*

$$U_n = \bigcup_{j=1}^{\infty} \left(x_j - \frac{1}{n2^j}, x_j + \frac{1}{n2^j} \right), \qquad U = \bigcap_{n=1}^{\infty} U_n.$$

Show that U is generic but has Lebesgue measure zero.

Proof. Each U_n is open. It is also dense because every nonempty open interval contains some rational x_j, hence meets U_n. Therefore $E_n := U_n^c$ is closed with empty interior, and thus nowhere dense. Consequently,

$$U^c = \bigcup_{n=1}^{\infty} E_n$$

is a countable union of nowhere dense sets, so U is generic.
For the measure estimate, the length of the jth interval in U_n is $2/(n2^j) = 1/(n2^{j-1})$, hence

$$m(U_n) \ \le \ \sum_{j=1}^{\infty} \frac{1}{n\,2^{j-1}} = \frac{1}{n}\sum_{k=0}^{\infty}\frac{1}{2^k} = \frac{2}{n}.$$

Since $U \subset U_n$ for all n, we have $m(U) \le \inf_n m(U_n) = 0$, so $m(U) = 0$. □

Theorem 23. *Every complete metric space X is of the second category in itself, that is, X cannot be written as the countable union of nowhere dense sets.*

Proof. For the sake of contradiction, suppose there are countably many nowhere dense sets $E_1, E_2, E_3, \ldots$ such that

$$X = \bigcup_{n=1}^{\infty} E_n.$$

Since $E_n \subseteq \overline{E_n}$ for all n, we have

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E_n}.$$

Since each E_n is nowhere dense, each $\overline{E_n}^c$ is an open dense subset of X. By De Morgan's Law,

$$\varnothing = X^c = \left(\bigcup_{n=1}^{\infty} \overline{E_n} \right)^c = \bigcap_{n=1}^{\infty} \overline{E_n}^c.$$

But each $\overline{E_n}^c$ is open and dense, so

$$\bigcap_{n=1}^{\infty} \overline{E_n}^c \neq \varnothing,$$

a contradiction. Therefore X cannot be written as a countable union of nowhere dense sets. □

Corollary 24. *In a complete metric space, a generic set is dense.*

Proof. Assume $E \subset X$ is generic but not dense. Then there exists a closed ball $\overline{B}$ entirely contained in E^c. Since E is generic, we can write $E^c = \cup_{n=1}^{\infty} F_n$ with each F_n nowhere dense, hence

$$\overline{B} = \bigcup_{n=1}^{\infty} (F_n \cap \overline{B}).$$

Each $F_n \cap \overline{B}$ is nowhere dense in $\overline{B}$ (with the relative topology). This contradicts the preceding theorem applied to the complete metric space $\overline{B}$. Hence E must be dense. □

4 Applications of the Baire Category Theorem

4.1 Continuity of the limit of a sequence of continuous functions

Suppose X is a complete metric space, $\{f_n\}$ is a sequence of continuous complex-valued functions on X, and

$$\lim_{n \to \infty} f_n(x) = f(x)$$

exists for each $x \in X$. It is well known that if the convergence is uniform in x, then the limit f is continuous. In general, with only pointwise convergence, we show that f has at least one point of continuity (in fact, a generic set of such points) by applying the category theorem.

Theorem 25. *Suppose that $\{f_n\}$ is a sequence of continuous complex-valued functions on a complete metric space X, and*

$$\lim_{n \to \infty} f_n(x) = f(x)$$

exists for every $x \in X$. Then the set of points where f is continuous is a generic set in X. In other words, the set of points where f is discontinuous is of the first category.

Therefore, f is in fact continuous at "most" points of X.

4.2 A localization lemma for pointwise limits

Lemma 26. *Suppose $\{f_n\}$ is a sequence of continuous functions on a complete metric space X, and $f_n(x) \to f(x)$ for each x as $n \to \infty$. Then, given an open ball $B \subset X$ and $\epsilon > 0$, there exists an open ball $B_0 \subset B$ and an integer $m \geq 1$ such that $|f_m(x) - f(x)| \leq \epsilon$ for all $x \in B_0$.*

Proof. Let Y be a closed ball contained in B. Then Y is a complete metric space. Define

$$E_\ell = \{x \in Y : \sup_{j,k>\ell} |f_j(x) - f_k(x)| \leq \epsilon\}.$$

Since $f_n(x)$ converges for every $x \in X$, we have

$$Y = \bigcup_{\ell=1}^{\infty} E_\ell. \tag{$*$}$$

Each E_ℓ is closed, being an intersection of sets of the form $\{x \in Y : |f_j(x) - f_k(x)| \leq \epsilon\}$, which are closed by continuity of f_j and f_k. Therefore, by the Baire Category Theorem applied to the complete metric space Y, some set in the union $(*)$, say E_m, contains a nonempty open set, hence an open ball B_0. By construction,

$$\sup_{j,k\geq m} |f_j(x) - f_k(x)| \leq \epsilon \qquad (x \in B_0),$$

and letting $k \to \infty$ yields $|f_m(x) - f(x)| \leq \epsilon$ for all $x \in B_0$. □

Proof of the Theorem.

Proof. Let D denote the set of discontinuities of f. For each $n \in \mathbb{N}$ let

$$F_n = \left\{x \in X : \operatorname{osc}(f)(x) \geq \tfrac{1}{n}\right\}.$$

Each F_n is closed (since $x \mapsto \operatorname{osc}(f)(x)$ is *upper* semicontinuous). Then $D = \bigcup_{n=1}^{\infty} F_n$. So it suffices to show that each F_n is nowhere dense. Suppose, to the contrary, that some F_n contains an open ball B.

By the lemma, taking $\epsilon = \frac{1}{4n}$, there exist $m \geq 1$ and an open ball $B_0 \subset B$ such that

$$|f_m(x) - f(x)| \leq \tfrac{1}{4n} \quad \text{for each } x \in B_0.$$

Since f_m is continuous, there exists an open ball $B' \subset B_0$ such that

$$|f_m(y) - f_m(z)| \leq \tfrac{1}{4n} \quad \text{for all } y, z \in B'.$$

Then, for all $y, z \in B'$,

$$|f(y) - f(z)| \leq |f(y) - f_m(y)| + |f_m(y) - f_m(z)| + |f_m(z) - f(z)| \leq \tfrac{3}{4n} < \tfrac{1}{n}.$$

Let x' denote the center of $B' \subset F_n$. Then $\operatorname{osc}(f)(x') < \frac{1}{n}$, which contradicts $x' \in F_n$. Hence F_n has empty interior. Since F_n is closed, its closure has empty interior as well, so F_n is nowhere dense. This proves that D is of the first category, i.e., the set of points of continuity of f is generic. □

4.3 Most continuous functions in $C[0,1]$ are nowhere differentiable

Theorem 27. *Let $C[0,1]$ be equipped with the supremum norm $\|\cdot\|_\infty$. The set $\mathcal{N} \subset C[0,1]$ of nowhere differentiable functions is residual (a dense G_δ). In particular, a generic continuous function on $[0,1]$ is nowhere differentiable.*

Proof. For $n, m \in \mathbb{N}$ and $q \in \mathbb{Q} \cap [0,1]$ define the open set

$$V_{n,m,q} := \bigcup_{\substack{s,t \in \mathbb{Q}\cap[0,1] \\ |s-q|<\frac{1}{2m},\, |t-q|<\frac{1}{2m},\, s\neq t}} \Big\{ f \in C[0,1] : \ \frac{|f(s)-f(t)|}{|s-t|} > n \Big\}.$$

Then set

$$\mathcal{U}_{n,m} := \bigcap_{q \in \mathbb{Q}\cap[0,1]} V_{n,m,q},$$

which is a G_δ set. We claim $\mathcal{U}_{n,m}$ is dense.

Fix $f_0 \in C[0,1]$ and $\varepsilon > 0$. Enumerate $\mathbb{Q} \cap [0,1] = \{q_k\}_{k\geq 1}$. For each k choose a rational interval

$$I_k = (a_k, b_k) \subset \Big(q_k - \tfrac{1}{2m},\, q_k + \tfrac{1}{2m}\Big), \quad a_k < b_k,\ a_k, b_k \in \mathbb{Q},$$

such that the intervals $\{I_k\}$ are pairwise disjoint and $\ell_k := b_k - a_k$ is so small that

$$\omega_{f_0}(\ell_k) := \sup_{|x-y|\leq \ell_k} |f_0(x) - f_0(y)| \leq \frac{\varepsilon}{2^{k+2}} \quad \text{and} \quad \ell_k \leq \frac{\varepsilon}{2^{k+2} n}.$$

Let $c_k = (a_k + b_k)/2 \in \mathbb{Q}$. Define a triangular (piecewise linear) bump $\phi_k \in C[0,1]$ supported in $[a_k, b_k]$ by

$$\phi_k(a_k) = \phi_k(b_k) = 0, \qquad \phi_k(c_k) = \alpha_k, \qquad \alpha_k := \frac{\varepsilon}{2^{k+1}}.$$

Set ϕ_k linear on $[a_k, c_k]$ and $[c_k, b_k]$. Then

$$\Big\| \sum_{k=1}^{\infty} \phi_k \Big\|_\infty \leq \sum_{k=1}^{\infty} \alpha_k = \varepsilon,$$

so $g := f_0 + \sum_{k\geq 1} \phi_k \in C[0,1]$ and $\|g - f_0\|_\infty \leq \varepsilon$.

For each fixed k, take $s_k = a_k$, $t_k = c_k$ (both rational). Since other bumps vanish on I_k and

$$|g(t_k) - g(s_k)| \;\geq\; |\phi_k(t_k) - \phi_k(s_k)| \;-\; |f_0(t_k) - f_0(s_k)| \;=\; \alpha_k - \omega_{f_0}(\ell_k),$$

while $|t_k - s_k| = \ell_k/2$, we have

$$\frac{|g(t_k) - g(s_k)|}{|t_k - s_k|} \;\geq\; \frac{\alpha_k - \omega_{f_0}(\ell_k)}{\ell_k/2} \;\geq\; \frac{\alpha_k/2}{\ell_k/2} \;=\; \frac{\alpha_k}{\ell_k} \;\geq\; n,$$

by the choices of ℓ_k and α_k. Also $|s_k - q_k|, |t_k - q_k| < \frac{1}{2m}$. Hence $g \in \mathcal{U}_{n,m}$. Since f_0 and ε were arbitrary, $\mathcal{U}_{n,m}$ is dense. (The same single-bump construction shows each $V_{n,m,q}$ is dense.)

Finally, set

$$\mathcal{U} := \bigcap_{n,m\in\mathbb{N}} \mathcal{U}_{n,m} = \bigcap_{n,m\in\mathbb{N}} \; \bigcap_{q\in\mathbb{Q}\cap[0,1]} V_{n,m,q},$$

a countable intersection of open sets. Since each $V_{n,m,q}$ is dense, $\mathcal{U}$ is residual by Baire. If $f \in \mathcal{U}$ were differentiable at some x, then there exist $M, \delta > 0$ with $|f(x+h) - f(x)| \leq M|h|$ for $|h| < \delta$. Choosing m with $1/m < \delta$ and $n > M$, the definition of $V_{n,m,q}$ (with rationals near x) would give s, t with $|s - x|, |t - x| < 1/m$ and $\frac{|f(s)-f(t)|}{|s-t|} > n$, a contradiction. Thus f is nowhere differentiable, and $\mathcal{N} \supset \mathcal{U}$ is residual. □

4.4 Open Mapping Theorem

Let X and Y be Banach spaces with norms $||\cdot||_X$ and $||\cdot||_Y$ respectively, and $T : X \to Y$ a mapping. Observe that T is continuous if and only if $T^{-1}(\mathcal{O}) = \{x \in X : T(x) \in \mathcal{O}\}$ is open in X whenever $\mathcal{O}$ is open in Y. This holds regardless of whether T is linear or not. In particular, if T has an inverse $S : Y \to X$ that is also continuous, the above observation applied to S shows that the image by T of any open set in X is open in Y. A mapping T that maps open sets to open sets is called an **open mapping.**

A bijective mapping has an inverse $T^{-1} : Y \to X$ defined as follows: if $y \in Y$, then $T^{-1}(y)$ is the unique element $x \in X$ so that $T(x) = y$. This definition is clear precisely because T is surjective and injective. In general, if T is linear, then the inverse T^{-1} is also linear, but T^{-1} need not be continuous. However, by the previous observation, we see that T^{-1} will be continuous if T is an open mapping. The next result says that surjectivity guarantees openness.

Theorem 28. *Suppose X and Y are Banach spaces, and $T : X \to Y$ is a continuous linear transformation. If T is surjective, then T is an open mapping.*

Proof. We denote by $B_X(x, r)$ and $B_Y(y, r)$ the open balls of radius r centered at $x \in X$ and $y \in Y$ respectively, and we denote by $B_X(r)$ and $B_Y(r)$ the open balls centered at the origin. Since T is linear, it suffices to show that $T(B_X(1))$ contains an open ball centered at the origin.

First, we prove the weaker statement that $\overline{T(B_X(1))}$ contains an open ball centered at the origin. Since T is surjective,

$$Y = \bigcup_{n=1}^{\infty} T(B_X(n)) = \bigcup_{n=1}^{\infty} \overline{T(B_X(n))}.$$

By the Baire Category Theorem, there exists n_0 such that $\overline{T(B_X(n_0))}$ has nonempty interior; hence there are $y_0 \in Y$ and $\varepsilon > 0$ with

$$B_Y(y_0, \varepsilon) \subset \overline{T(B_X(n_0))}.$$

By linearity, $T(B_X(n_0)) = T(n_0 B_X(1)) = n_0\, T(B_X(1))$, so applying the continuous scaling map $y \mapsto y/n_0$ yields

$$B_Y\Big(\tfrac{y_0}{n_0}, \tfrac{\varepsilon}{n_0}\Big) \subset \overline{T(B_X(1))}.$$

Choose $y_1 = T(x_1) \in T(B_X(1))$ with $\|y_1 - \frac{y_0}{n_0}\|_Y < \frac{\varepsilon}{2n_0}$. Then

$$B_Y\Big(0, \tfrac{\varepsilon}{2n_0}\Big) \subset B_Y\Big(y_1, \tfrac{\varepsilon}{2n_0}\Big) - y_1 \subset \overline{T(B_X(1))} - T(B_X(1)) \subset \overline{T(B_X(2))}.$$

(The last inclusion follows since differences of limits of points in $T(B_X(1))$ are limits of images $T(u - v)$ with $u, v \in B_X(1)$, hence $u - v \in B_X(2)$.) By homogeneity, we get

$$B_Y\Big(0, \tfrac{\varepsilon}{4n_0}\Big) \subset \overline{T(B_X(1))}.$$

Scaling T if necessary (replace T by $\frac{4n_0}{\varepsilon} T$), we may assume

$$B_Y(1) \subset \overline{T(B_X(1))} \quad \text{and hence} \quad B_Y(2^{-k}) \subset \overline{T(B_X(2^{-k}))} \text{ for all } k \in \mathbb{N}.$$

Next, we strengthen this to show

$$B_Y\big(\tfrac{1}{2}\big) \subset T(B_X(1)).$$

Let $y \in B_Y(\frac{1}{2})$. Using $B_Y(2^{-1}) \subset \overline{T(B_X(2^{-1}))}$, pick $x_1 \in B_X(2^{-1})$ with $\|y - T(x_1)\|_Y < 2^{-2}$. Then using $B_Y(2^{-2}) \subset \overline{T(B_X(2^{-2}))}$, choose $x_2 \in B_X(2^{-2})$ with $\|y - T(x_1) - T(x_2)\|_Y < 2^{-3}$, and continue inductively to obtain $x_k \in B_X(2^{-k})$ such that

$$\Big\| y - \sum_{j=1}^{k} T(x_j) \Big\|_Y < 2^{-(k+1)}.$$

Since $\sum_{j=1}^{\infty} \|x_j\|_X \leq \sum_{j=1}^{\infty} 2^{-j} = 1$, the series $\sum_{j=1}^{\infty} x_j$ converges in X to some x with $\|x\|_X \leq 1$. By continuity of T,

$$T(x) = \sum_{j=1}^{\infty} T(x_j) = y.$$

Thus $y \in T(B_X(1))$, proving $B_Y(\frac{1}{2}) \subset T(B_X(1))$, and hence $T(B_X(1))$ contains an open ball about 0. Therefore T is open. □

There are two interesting corollaries of this theorem.

Corollary 29. *If X and Y are Banach spaces, and $T : X \to Y$ is a continuous bijective linear transformation, then the inverse $T^{-1} : Y \to X$ is also continuous. Hence there exist constants $c, C > 0$ with*

$$c||x||_X \leq ||T(x)||_Y \leq C||x||_X \quad \textit{for all } x \in X.$$

This provides another very useful application of the Open Mapping Theorem, which gives us an easy way to check the equivalence of two norms.

Corollary 30. *Suppose the vector space V is equipped with two norms $|| \cdot ||_1$ and $|| \cdot ||_2$. If*

$$||v||_1 \leq C||v||_2 \textit{ for all } v \in V,$$

and V is complete with respect to both norms, then $|| \cdot ||_1$ and $|| \cdot ||_2$ are equivalent.

We recall that a mapping $T : X \to Y$ is **surjective** if $T(X) = Y$, and **injective** if $T(x) = T(y)$ implies $x = y$. Also, T is **bijective** if it is both surjective and injective.

4.5 Uniform Boundedness Theorem

Theorem 31 (Banach–Steinhaus)**.** *Let (T_n) be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed space Y such that for every $x \in X$ the sequence $(\|T_n x\|)$ is bounded, i.e.,*

$$\|T_n x\| \leq c_x, \qquad n = 1, 2, \ldots \tag{1}$$

for some (possibly x-dependent) constant c_x. Then the sequence of operator norms $(\|T_n\|)$ is bounded; that is, there exists $C < \infty$ such that

$$\|T_n\| \leq C, \qquad n = 1, 2, \ldots. \tag{2}$$

Equivalently, $\|T_n x\| \leq C\|x\|$ for all $x \in X$ and all n.

Proof. For every $k \in \mathbb{N}$, let

$$A_k := \{x \in X : \ \|T_n x\| \le k \text{ for all } n\}.$$

Each A_k is closed: if $x_j \to x$ with $x_j \in A_k$, then for each fixed n, $\|T_n x\| = \lim_{j\to\infty} \|T_n x_j\| \le k$. By (1), every $x \in X$ belongs to some A_k, hence

$$X = \bigcup_{k=1}^{\infty} A_k.$$

Since X is complete, Baire's theorem implies that some A_{k_0} has nonempty interior; in particular, there exist $x_0 \in X$ and $r > 0$ with

$$B_0 := B(x_0; r) \subset A_{k_0}. \tag{3}$$

Fix $x \in X$, $x \neq 0$, and set

$$z := x_0 + \gamma x, \qquad \gamma := \frac{r}{2\|x\|}. \tag{4}$$

Then $\|z - x_0\| = \gamma \|x\| = r/2 < r$, so $z \in B_0$. From (3) we have $\|T_n z\| \le k_0$ and $\|T_n x_0\| \le k_0$ for all n. By (4),

$$x = \frac{1}{\gamma}(z - x_0),$$

hence, for all n,

$$\|T_n x\| = \frac{1}{\gamma}\,\|T_n(z - x_0)\| \le \frac{1}{\gamma}\big(\|T_n z\| + \|T_n x_0\|\big) \le \frac{1}{\gamma}\,(2k_0) = \frac{4k_0}{r}\,\|x\|.$$

Therefore

$$\|T_n\| = \sup_{\|x\|=1} \|T_n x\| \le \frac{4k_0}{r} \qquad \text{for all } n,$$

which is (2) with $C = 4k_0/r$. □

Corollary 32 (Functional version on a set of second category)**.** *Let B be a Banach space and $L \subset B^*$ a family of bounded linear functionals. Suppose there exists a set $E \subset B$ of second category such that*

$$\sup_{\ell \in L} |\ell(f)| < \infty \quad \textit{for all } f \in E.$$

Then $\sup_{\ell \in L} \|\ell\| < \infty$; i.e., L is uniformly bounded.

Proof. For $M \in \mathbb{N}$ set

$$E_M := \{f \in B : \sup_{\ell \in L} |\ell(f)| \leq M\} = \bigcap_{\ell \in L} \{f \in B : |\ell(f)| \leq M\},$$

so each E_M is closed. Since $E = \bigcup_{M \in \mathbb{N}} E_M$ and E is of the second category, some E_{M_0} has nonempty interior; thus there exist $f_0 \in B$ and $r_0 > 0$ with $B(f_0; r_0) \subset E_{M_0}$. Fix $0 < r < r_0$. For any $g \in B$ with $\|g\| \leq r$, we have $f := g + f_0 \in B(f_0; r_0) \subset E_{M_0}$, hence for all $\ell \in L$,

$$|\ell(g)| = |\ell(f) - \ell(f_0)| \leq |\ell(f)| + |\ell(f_0)| \leq 2M_0.$$

Taking the supremum over $\{g : \|g\| \leq r\}$ yields $\|\ell\| \leq 2M_0/r$ for every $\ell \in L$, so $\sup_{\ell \in L} \|\ell\| < \infty$. □

4.6 The Closed Graph Theorem

Let X and Y be metric spaces. Then a function $f : X \to Y$ is said to have a closed graph if

$$\text{graph}(f) = \{(x, f(x)) : x \in X\}$$

is a closed subset of the product space $X \times Y$. Note in particular that if f is bijective, then f has a closed graph if and only if f^{-1} does.
To compare closed graph to the related property of continuity, we consider the following three statements concerning a sequence (x_n) in X and elements, $x \in X, y \in Y$:

(a) $x_n \to x$,

(b) $f(x_n) \to y$,

(c) $y = f(x)$.

Continuity is the statement that (a) implies (b) and (c). Having a closed graph is the statement that (a) and (b) together imply (c).

Theorem 33 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear map. Then T is continuous if and only if it has a closed graph.*

Proof. Every continuous map has a closed graph, so we prove the converse. This will be a simple application of the open mapping theorem.
Since X and Y are Banach spaces, the product space $X \times Y$ is a Banach space when equipped with the norm

$$||(x, y)||_{X \times Y} = ||x||_X + ||y||_Y.$$

Since graph(T) is a closed linear subspace of $X \times Y$, it is a Banach space in its own right.
Consider the projection $\Pi_1 : \text{graph}(T) \to X$ defined by $\Pi_1(x, y) = x$.
Clearly, Π_1 is a bijection of graph(T) onto X. Moreover,

$$||\Pi_1(x, y)||_X = ||x||_X \leq ||(x, y)||_{X \times Y},$$

so Π_1 is continuous. By the Open Mapping Theorem, Π_1^{-1} is a continuous linear map of X into graph(T). On the other hand, the projection $\Pi_2 : \text{graph}(T) \to Y$ defined by $\Pi_2(x, y) = y$ is also continuous since

$$||\Pi_2(x, y)||_Y = ||y||_Y \leq ||(x, y)||_{X\times Y}.$$

It follows that $T = \Pi_2 \circ \Pi_1^{-1}$ is a continuous map of X into Y. □

Example 34. ***(Grothendieck's theorem on closed subspaces of L^p)***

As an application of the closed graph theorem, we prove the following result:

Theorem 35. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space, that is, $\mu(X) < \infty$. Suppose that*

(a) E is a closed subspace of $L^p(X, \mu)$ for some $1 \leq p < \infty$, and

(b) E is contained in $L^\infty(X, \mu)$.

Then E is finite dimensional.

Since $E \subset L^\infty$ and X has finite measure, we find that $E \subset L^2$ with

$$\|f\|_{L^2} \leq \mu(X)^{1/2} \, \|f\|_{L^\infty} \qquad \text{whenever } f \in E.$$

The essential idea in the proof of the theorem is to reverse this inequality, and then use the Hilbert space structure of L^2.
Equipped with the L^p-norm, E is a Banach space since it is a closed subspace of $L^p(X, \mu)$.
Let

$$I : E \to L^\infty(X, \mu)$$

denote the identity mapping $I(f) = f$. Then I is linear and has closed graph: indeed, if $f_n \to f$ in E and $f_n \to g$ in L^∞, there exists a subsequence of $\{f_n\}$ that converges almost everywhere to f, while L^∞-convergence implies almost-everywhere convergence to g along a subsequence; hence $f = g$ a.e. By the Closed Graph Theorem there is an $M > 0$ so that

$$\|f\|_{L^\infty} \leq M \, \|f\|_{L^p} \qquad \text{for all } f \in E. \tag{1}$$

Lemma 36. *Under the assumptions of the theorem, there exists $A > 0$ so that*

$$\|f\|_{L^\infty} \leq A\|f\|_{L^2} \quad \textit{for all } f \in E.$$

Proof. If $1 \leq p \leq 2$, then Hölder's inequality with conjugate exponents $r = 2/p$ and $r^* = 2/(2 - p)$ yields

$$\int |f|^p \leq \left(\int |f|^2\right)^{p/2} \left(\int 1\right)^{\frac{2-p}{2}}.$$

Since X has finite measure, after taking pth roots we obtain a constant $B > 0$ such that $\|f\|_{L^p} \le B\|f\|_{L^2}$ for all $f \in E$. Together with (1), this gives

$$\|f\|_{L^\infty} \le M\|f\|_{L^p} \le MB\|f\|_{L^2},$$

proving the claim when $1 \le p \le 2$.
When $2 < p < \infty$, note first that $|f(x)|^p \le \|f\|_{L^\infty}^{p-2}|f(x)|^2$, and integrating gives

$$\|f\|_{L^p}^p \le \|f\|_{L^\infty}^{p-2}\,\|f\|_{L^2}^2.$$

Using (1), $\|f\|_{L^\infty} \le M\|f\|_{L^p}$, hence

$$\|f\|_{L^p}^p \le M^{p-2}\|f\|_{L^p}^{p-2}\|f\|_{L^2}^2,$$

so (if $\|f\|_{L^p} = 0$ the claim is trivial; otherwise) dividing by $\|f\|_{L^p}^{p-2}$ yields

$$\|f\|_{L^p} \le M^{\frac{p-2}{2}}\|f\|_{L^2}.$$

Combining again with (1) gives

$$\|f\|_{L^\infty} \le M\|f\|_{L^p} \le M^{\frac{p}{2}}\|f\|_{L^2}.$$

Thus the lemma holds with $A = MB$ for $p \le 2$ and $A = M^{p/2}$ for $p > 2$. □

We now return to the proof of the theorem. Suppose $f_1, \dots, f_n$ is an orthonormal set in L^2 of functions in E, and let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^n$,

$$\mathbb{B} = \{\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |\zeta_j|^2 \le 1\}.$$

For each $\zeta \in \mathbb{B}$, let $f_\zeta(x) = \sum_{j=1}^n \zeta_j f_j(x)$. By orthonormality,

$$\|f_\zeta\|_{L^2}^2 = \sum_{j=1}^{n} |\zeta_j|^2 \le 1,$$

and the lemma gives $\|f_\zeta\|_{L^\infty} \le A$. Hence for each ζ, there exists a measurable set X_ζ of full measure in X such that

$$|f_\zeta(x)| \le A \quad \text{for all } x \in X_\zeta. \tag{2}$$

Let $\{\zeta^{(k)}\}_{k\ge1}$ be a countable dense subset of $\mathbb{B}$ and set $X' = \bigcap_{k\ge1} X_{\zeta^{(k)}}$, which still has full measure. For fixed $x \in X'$, the map $\zeta \mapsto f_\zeta(x)$ is continuous, so from (2) for all $\zeta^{(k)}$ we deduce

$$|f_\zeta(x)| \le A \quad \text{for all } x \in X' \text{ and all } \zeta \in \mathbb{B}. \tag{3}$$

From this we claim that

$$\sum_{j=1}^{n} |f_j(x)|^2 \le A^2 \quad \text{for all } x \in X'. \tag{4}$$

Indeed, if the left-hand side is nonzero, set $\sigma = \left(\sum_{j=1}^{n} |f_j(x)|^2\right)^{1/2}$ and $\zeta_j = \overline{f_j(x)}/\sigma$. Then by (3),

$$\frac{1}{\sigma}\sum_{j=1}^{n} |f_j(x)|^2 = |f_\zeta(x)| \le A,$$

so $\sigma \le A$, proving (4). Finally, integrating (4) over X' and using orthonormality of $\{f_1, \ldots, f_n\}$ in L^2 gives

$$n = \sum_{j=1}^{n} \|f_j\|_{L^2}^2 \le A^2\,\mu(X),$$

so $n \le A^2\,\mu(X)$. Therefore E is finite dimensional.

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