A Discussion on Analytic Web Geometry

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Abstract

This report surveys analytic web geometry. We explain how plane curves give rise to algebraic planar webs and how Abel's theorem produces Abelian relations for these webs. We later review work on exceptional webs and show how functional identities yield polylogarithmic webs.

Introduction

Web geometry studies configurations of foliations and the constraints imposed on them by their *Abelian relations*. In the local analytic planar setting, a d-web $\mathcal{W}(d)$ on $(\mathbb{C}^2, 0)$ is a collection of d germs of foliations in general position (i.e., pairwise transverse at the origin), given by the level sets of holomorphic functions $F_1, \ldots, F_d$.

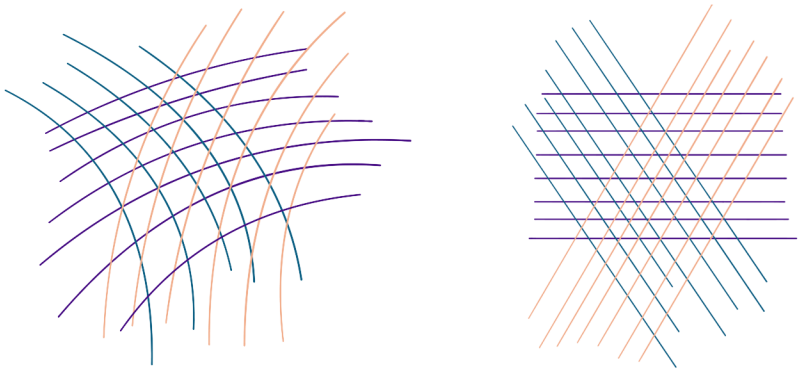


Figure 1: A 3-web on $(\mathbb{C}^2, 0)$ (left) and a linear 3-web (right) [2].

Following the classical work of Blaschke, Bol, and Chern, and the modern expositions of Griffiths and Hénaut [2–4], a fundamental invariant of such a web is its *rank* $\mathrm{rk}\mathcal{W}(d)$. This is the dimension of its space of Abelian relations. Bol's inequality $\mathrm{rk}\mathcal{W}(d) \leq \frac{(d-1)(d-2)}{2}$ gives a universal upper bound. Webs that attain this bound are said to have *maximal rank*.

Many examples in this subject come from algebraic geometry. To a reduced plane curve $C \subset \mathbb{P}^2$ of degree d, we can associate an algebraic d-web $\mathcal{L}_C(d)$ on the dual projective plane $\check{\mathbb{P}}^2$. Abel's theorem identifies the space of Abelian relations of $\mathcal{L}_C(d)$ with the space of Abelian differentials $H^0(C, \omega_C)$. This shows that algebraic planar webs have maximal rank and attain Bol's bound. Conversely, the Lie-Darboux-Griffiths theorem gives an algebraization criterion: a linear web $\mathcal{L}(d)$ with an algebraizing Abelian relation must be algebraic. In particular, any linearizable planar web of maximal rank comes from a plane curve [2, 4]. This leads to the main idea of an *exceptional* planar web: a maximal-rank web that is not algebraizable.

Exceptional webs first appear in degree 5. Bol constructed a planar 5-web B of maximal rank that is not algebraizable; this gave a counterexample to the old expectation that all maximal-rank webs should be algebraic. Hénaut showed that the Abelian relations of B can be described by functional equations for the logarithm and dilogarithm, and introduced polylogarithmic webs associated to higher-order polylogarithms [2]. More recently, Pirio developed an ODE/functional-equation approach to Abelian relations and used it to construct new exceptional planar 5-webs. He answered Chern's question about the uniqueness of Bol's example [1].

In this report we give a brief introduction to these ideas and organize the material as follows.

We begin by introducing the basic definitions, and we define exceptional planar webs in Section 1. In Section 2, we explain the relationship between plane curves and algebraic webs of maximal rank and elaborate on Abel's theorem. In Section 3, we focus on 3-webs. We recall Blaschke's curvature form, give the characterization of hexagonal 3-webs, and present the logarithmic hexagonal example. With more terminology introduced, in Section 4 we review Hénaut's linearization theory, which includes his polynomial $P_{\mathcal{W}}$ and Poincaré's theorem on maximal-rank 4-webs. In Section 5, we describe how to eliminate variables in the functional equation that defines Abelian relations, and how to obtain explicit PDE conditions for planar 5-webs with a parallelizable 4-subweb. Finally, in Section 6, we discuss exceptional planar 5-webs and related topics.

1 Planar webs, Abelian relations, and Bol's bound

In this section, we mainly focus on introducing the general definitions and notation [2] that will be used throughout the paper.

To begin, let $\mathcal{O} := \mathbb{C}\{x, y\}$ be the ring of convergent power series in two variables.

1.1 Abelian relations and rank

Definition 1.1. A *d-web* on $(\mathbb{C}^2, 0)$ is a collection $\mathcal{W} = \mathcal{W}(F_1, \dots, F_d)$ of holomorphic functions $F_i \in \mathcal{O}$ with $F_i(0) = 0$ such that

$$\mathrm{d}F_i(0) \wedge \mathrm{d}F_j(0) \neq 0 \quad \text{for } 1 \leq i < j \leq d.$$

The leaves of the i-th foliation are the level sets $\{F_i = \mathrm{const}\}$.

Remark 1.2. For simplicity, throughout, a *planar* web will mean a germ of a web on $(\mathbb{C}^2, 0)$ in the sense of Definition 1.1.

Definition 1.3. Two webs are *equivalent* if there exists a local biholomorphism of $(\mathbb{C}^2, 0)$ sending the foliations of one to those of the other.

Remark 1.4. We will freely write $\mathcal{W}(F_1, \dots, F_d)$, or simply $\mathcal{W}(d)$, for the germ of such a web at the origin.

The main invariant of $\mathcal{W}(d)$ is related to the notion of an Abelian relation.

Definition 1.5. Consider the web $\mathcal{W}(d)$. An *Abelian relation* is a d-tuple $(g_1(F_1), \dots, g_d(F_d))$ with $g_i \in \mathbb{C}\{z\}$ such that $\sum_{i=1}^{d} g_i(F_i)\mathrm{d}F_i = 0$. The space of such relations,

$$\mathcal{A}(d) := \Big\{ (g_i(F_i))_{1 \leq i \leq d} \Big| g_i \in \mathbb{C}\{z\}, \sum_i g_i(F_i)\mathrm{d}F_i = 0 \Big\},$$

is a finite-dimensional $\mathbb{C}$-vector space; its dimension $\mathrm{rk}\mathcal{W}(d) := \dim_{\mathbb{C}} \mathcal{A}(d)$ is the *rank* of the web. For a given web $\mathcal{W}$, we will also write $\mathcal{A}(\mathcal{W})$ or $\mathcal{A}[\mathcal{W}]$ for its space of Abelian relations.

Theorem 1.6. *For every d-web $\mathcal{W}(d)$ on $(\mathbb{C}^2, 0)$, we have* $\mathrm{rk}\mathcal{W}(d) \leq \frac{(d-1)(d-2)}{2}$.

This optimal inequality goes back to Bol. Hénaut reviews both the classical Poincaré-Blaschke proof and a holonomic $\mathcal{D}$-module proof; for simplicity we omit the proof, and both approaches are described in [2].

Definition 1.7. A planar d-web is said to have *maximal rank* if equality holds in Theorem 1.6.

Definition 1.8. A d-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$ is *exceptional* if

1. it has maximal rank (Definition 1.7); and
2. it is not algebraizable.

Remark 1.9. Here "algebraizable" (which we will discuss more clearly in Section 2) means that $\mathcal{W}$ is equivalent to an algebraic planar web $\mathcal{L}_C$ associated to some plane curve $C \subset \mathbb{P}^2$. This is the planar case $n = 1$ of the general exceptional web $\mathcal{E}(d, 2, n)$ of codimension n in $(\mathbb{C}^{2n}, 0)$ introduced in [2].

In Section 2 we will see that webs arising from plane curves are algebraic models that attain Bol's bound.

2 Algebraic planar webs and Abel's theorem

2.1 From plane curves to linear webs

We review some terminology from [2] to understand the main technical construction in this subsection.

Let $C \subset \mathbb{P}^2$ be a *reduced* algebraic curve of degree d. It may be reducible and singular. A generic line $l(0) \in G(1, \mathbb{P}^2) = \check{\mathbb{P}}^2$ meets C in d distinct smooth points $p_i(0)$ which, by *duality*, determine d distinct lines in $\check{\mathbb{P}}^2$ passing through $l(0)$.

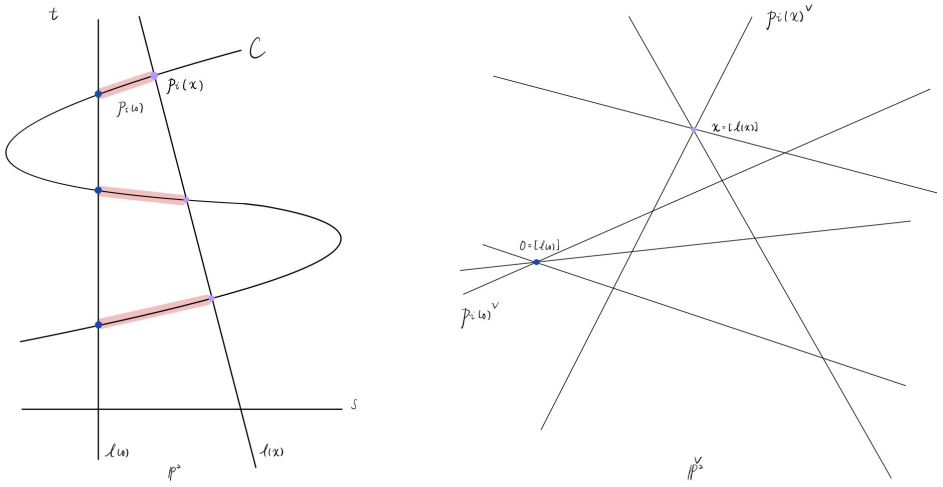


Figure 2: Construction of the algebraic web $\mathcal{L}_C(d)$ (right) associated to a plane curve $C \subset \mathbb{P}^2$ (left) [2]. For $x \in \check{\mathbb{P}}^2$, the corresponding $l(x)$ meets C in points $p_1(x), \ldots, p_d(x)$, which determine the d foliations of the web on the dual plane.

This construction extends to lines $l(x)$ near $l(0)$ to obtain a d-web $\mathcal{L}_C(d)$ on $(\mathbb{C}^2, 0) = (\check{\mathbb{P}}^2, l(0))$, which is *linear* because its leaves are straight lines (not necessarily parallel).

If x is near $0 \in \mathbb{C}^2$, the leaves passing through x correspond, by duality, to the points $p_i(x)$ satisfying

$$l(x) \cap C = \sum_{i=1}^{d} p_i(x)$$

as 0-cycles on C. We call $\mathcal{L}_C(d)$ the algebraic web associated to $C \subset \mathbb{P}^2$.

If C contains no straight line, the leaves of $\mathcal{L}_C(d)$ are the tangent lines to the dual curve $\check{C} \subset \check{\mathbb{P}}^2$; otherwise, the corresponding foliations are given by the associated pencils of lines.

The following definition recaps the notions introduced above and records the corresponding algebraic terminology.

Definition 2.1. A planar web on $(\mathbb{C}^2, 0)$ is *linear* if it is defined by straight lines (not necessarily parallel), and *linearizable* if it is equivalent to a linear web. A planar web of the form $\mathcal{L}_C(d)$ as above is called *algebraic*. A web is *algebraizable* if it is equivalent to an algebraic one.

2.2 Abel's theorem

We now explain how Abel's theorem produces Abelian relations for the web $\mathcal{L}_C(d)$, and why algebraic webs have maximal rank. Some of the arguments in this subsection are inspired by Griffiths' exposition in [4]. I also incorporate some discussion from class notes [6, 7].

Keep the curve $C \subset \mathbb{P}^2$ of degree d and the associated algebraic web $\mathcal{L}_C(d)$ on $(\mathbb{C}^2, 0) = (\check{\mathbb{P}}^2, l(0))$ as above. Consider

$$I = \{(p, l) \in C \times \check{\mathbb{P}}^2 \mid p \in C \cap l\}$$

with projections $q : I \to C$ and $\pi : I \to \check{\mathbb{P}}^2$. For $\omega \in H^0(C, \omega_C)$ (where ω_C is the dualizing sheaf of C), its *trace* along the lines of $\check{\mathbb{P}}^2$ is the meromorphic 1-form

$$\mathrm{Trace}(\omega) := \pi_*(q^*\omega),$$

defined on the open subset of $\check{\mathbb{P}}^2$ where the line $l(x)$ meets C in d distinct smooth points.

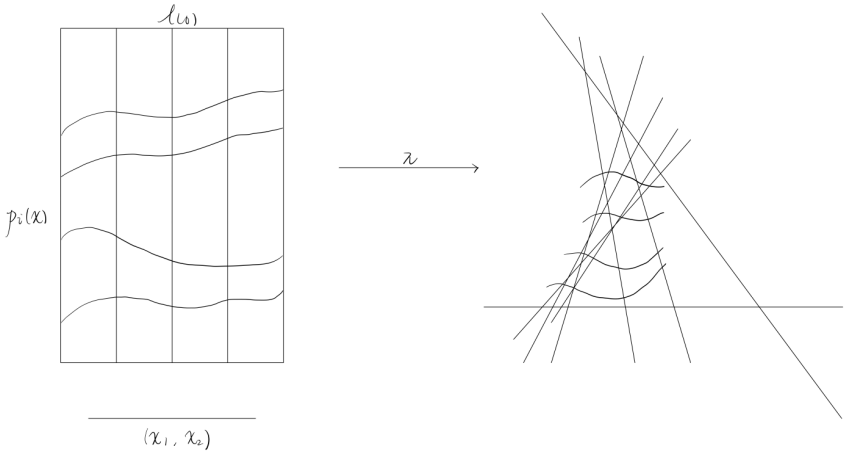


Figure 3: This is a local picture of the incidence correspondence I and the projection $\pi : I \to \check{\mathbb{P}}^2$ near $l(0)$. Over a neighborhood of lines $l(x)$ meeting C transversely, π is a finite covering with local sheets $p = p_i(x)$ corresponding to the intersection points $p_i(x) \in C \cap l(x)$. $\mathrm{Trace}(\omega) = \pi_*(q^*\omega)$ is obtained by summing the pullbacks $p_i^*\omega$ over these sheets [4].

From now on in this subsection, we assume that C is smooth. In particular, we may identify the dualizing sheaf ω_C with the sheaf Ω^1_C of holomorphic 1-forms, and we write $M = C$ for the corresponding compact Riemann surface of genus g.

Write $\Omega^1(M) := H^0(M, \Omega^1_M)$ for the space of holomorphic 1-forms on M. Fix a symplectic basis $\{\gamma_1, \dots, \gamma_{2g}\} \subset H_1(M, \mathbb{Z})$ for the intersection pairing. Choose a basis $\{\omega_1, \dots, \omega_g\} \subset \Omega^1(M)$ normalized so that $\int_{\gamma_i} \omega_j = \delta_{ij}$ for $1 \le i, j \le g$. With this choice, the period matrix has the form $\Pi = \begin{pmatrix} I_g & Z \end{pmatrix}$, where $Z = Z^t$ and $\Im(Z) > 0$, and the Jacobian of M is the complex torus

$$J(M) := \Omega^1(M)^\vee / H_1(M, \mathbb{Z}) \simeq \mathbb{C}^g / \Lambda_M,$$

where Λ_M is the lattice generated by the columns of Π.

Let $\mathrm{Div}(M)$ be the group of divisors on M, and let $\mathrm{Div}^0(M) := \{D \in \mathrm{Div}(M) \mid \deg D = 0\}$ be the subgroup of degree-zero divisors. Fix a basepoint $p_0 \in M$. For a divisor $D = \sum_i n_i [P_i] \in \mathrm{Div}^0(M)$, the Abel-Jacobi map is

$$AJ(D) := \Big(\sum_i n_i \int_{p_0}^{P_i} \omega_1, \dots, \sum_i n_i \int_{p_0}^{P_i} \omega_g \Big) \in \mathbb{C}^g / \Lambda_M \simeq J(M).$$

Recall that the Abel-Jacobi map $AJ : \mathrm{Div}^0(M) \to J(M)$ is well defined (independent of the choice of paths) and is a group homomorphism, and that $AJ((f)) = 0$ for any meromorphic function $f \in K(M)^*$. Abel's theorem then states that

$$AJ(D) = 0 \quad \Longleftrightarrow \quad D = (f) \text{ for some } f \in K(M)^*.$$

By the Riemann-Roch theorem, we have $\dim_{\mathbb{C}} \Omega^1(M) = g$ for a compact Riemann surface M of genus g. For a smooth plane curve $C \subset \mathbb{P}^2$ of degree d, this genus is $g = \frac{(d-1)(d-2)}{2}$.

We now reinterpret the trace construction in terms of the Abel-Jacobi map. For $x \in (\check{\mathbb{P}}^2, l(0))$ near 0, recall that $l(x) \cap C = \{p_1(x), \dots, p_d(x)\}$, with each $p_i(x)$ depending holomorphically on x. Since C is smooth, we identify it with its normalization M, and regard the points $p_i(x)$ as points of M. Fix $x_0 = 0$ and set $p_i^0 := p_i(x_0)$.

We define a degree-zero divisor on M by

$$D(x) := \sum_{i=1}^d [p_i(x)] - \sum_{i=1}^d [p_i^0] \in \mathrm{Div}^0(M).$$

Let L_x, L_0 be linear forms on $\mathbb{P}^2$ whose zero sets are $l(x)$ and $l(0)$, respectively. The meromorphic function $f_x := \frac{L_x}{L_0}$ on $\mathbb{P}^2$ restricts to a meromorphic function $f_x|_C$ on C, whose divisor is

$$(f_x|_C) = (l(x) \cdot C) - (l(0) \cdot C) = \sum_{i=1}^d [p_i(x)] - \sum_{i=1}^d [p_i^0] = D(x),$$

which counts intersection multiplicities. Thus $D(x) = (f_x|_C)$ is principal for every x, so Abel's theorem implies $AJ(D(x)) = 0 \in J(M)$ for all x.

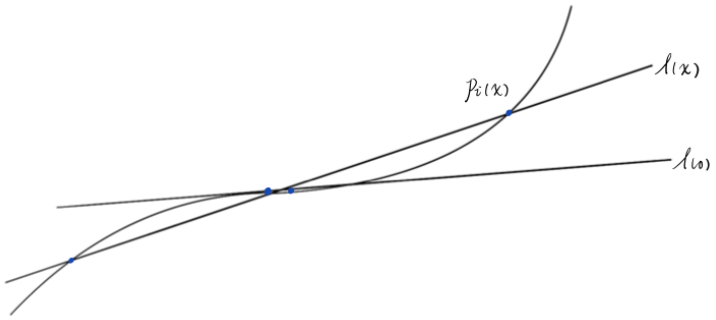


Figure 4: $l(0)$ and $l(x)$ intersect the plane curve $C \subset \mathbb{P}^2$, defining the intersection divisors $(l(0) \cdot C)$ and $(l(x) \cdot C)$ on C. Their difference $D(x) := (l(x) \cdot C) - (l(0) \cdot C)$ is a principal divisor [4].

Using the same basepoint $p_0 \in M$ and the basis $\{\omega_1, \ldots, \omega_g\}$ of $\Omega^1(M)$ as above, additivity of AJ gives

$$AJ(D(x)) = \sum_{i=1}^{d} AJ\big([p_i(x)] - [p_i^0]\big) = \Big(\sum_i \int_{p_i^0}^{p_i(x)} \omega_1, \ldots, \sum_i \int_{p_i^0}^{p_i(x)} \omega_g\Big) \in J(M).$$

Since $AJ(D(x)) \equiv 0$ in $J(M)$, this map is constant, so its derivative is zero.

For convenience, set

$$A_k(x) := \sum_{i=1}^{d} \int_{p_i^0}^{p_i(x)} \omega_k, \quad 1 \le k \le g.$$

Then each A_k is locally a holomorphic function on $(\check{\mathbb{P}}^2, l(0))$ (with values in $\mathbb{C}$), and the class of $(A_1(x), \ldots, A_g(x))$ modulo the period lattice Λ_M is identically zero.

Now consider the Abel-Jacobi map $u_q : M \longrightarrow J(M)$, $p \mapsto AJ([p] - [q])$, for some $q \in M$. Its differential

$$du_q(p) : T_pM \longrightarrow T_{u_q(p)}J(M) \simeq \mathbb{C}^g$$

is given by

$$du_q(p)(v) = \big(\omega_1(p)(v), \ldots, \omega_g(p)(v)\big)$$

for every $v \in T_pM$. Since $AJ(D(x)) = \sum_i AJ([p_i(x)] - [p_i^0]) = \sum_i u_{p_i^0}\big(p_i(x)\big)$, its differential at $x = 0$ in the direction of $w \in T_0(\check{\mathbb{P}}^2)$ is

$$\begin{aligned} \mathrm{d}AJ(D(x))\big|_{x=0}(w) &= \sum_{i=1}^{d} du_{p_i^0}\big(p_i^0\big)\big((\mathrm{d}p_i)_{x=0}(w)\big) \\ &= \Big(\sum_i \omega_1\big(p_i^0\big)\big((\mathrm{d}p_i)_{x=0}(w)\big), \ldots, \sum_i \omega_g\big(p_i^0\big)\big((\mathrm{d}p_i)_{x=0}(w)\big)\Big) \\ &= 0 \in T_0J(M) \simeq \mathbb{C}^g \quad \text{for all } w \in T_0(\check{\mathbb{P}}^2). \end{aligned}$$

Since $AJ(D(x)) \equiv 0$ in $J(M)$, each component must vanish:

$$\sum_{i=1}^{d} \omega_k\big(p_i^0\big)\big((\mathrm{d}p_i)_{x=0}(w)\big) = 0 \quad \text{for every } w \in T_0(\check{\mathbb{P}}^2),\ 1 \le k \le g. \tag{1}$$

For a fixed holomorphic differential $\omega \in \Omega^1(M)$, define $\mathrm{Trace}(\omega) := \pi_*(q^*\omega)$ as above, so that $\mathrm{Trace}(\omega)_x = \sum_{i=1}^d (p_i^*\omega)_x$. Equivalently, for $w \in T_x(\check{\mathbb{P}}^2)$,

$$\begin{aligned}\mathrm{Trace}(\omega)_x(w) &= \sum_{i=1}^{d} \omega\big(p_i(x)\big)\big((\mathrm{d}p_i)_x(w)\big) \\ &= \sum_{i=1}^{d} \omega_k\big(p_i^0\big)\big((\mathrm{d}p_i)_0(w)\big) \qquad (\text{at } x=0,\ \omega=\omega_k) \\ &= 0 \qquad (\text{by } (1)).\end{aligned}$$

Since the choice of x_0 (hence of $x = 0$) was arbitrary, this shows that $\mathrm{Trace}(\omega_k) \equiv 0$ on the open subset where $l(x)$ meets C transversely in d distinct points. By linearity in ω, we conclude that $\mathrm{Trace}(\omega) = 0$ for every $\omega \in H^0(C, \omega_C) \simeq \Omega^1(M)$.

To find an Abelian relation from this identity, we first write everything in local coordinates. Near each point p_i^0, choose affine coordinates (s, t) on $\mathbb{P}^2$ in which C is given locally by $t = \xi_i(s)$ and $l(x)$ is given by $s = F_i$. On the corresponding branch of C we can write $\omega|_C = g_i(s)\mathrm{d}s$, so that along $l(x)$ we have $p_i^*\omega = g_i(F_i)\,\mathrm{d}F_i$. Thus the identity $\mathrm{Trace}(\omega) = 0$, i.e. $\sum_{i=1}^d p_i^*\omega = 0$, becomes

$$\sum_{i=1}^{d} g_i(F_i)\mathrm{d}F_i = 0,$$

which is an Abelian relation in the sense of Definition 1.5.

This construction gives a linear map

$$H^0(C, \omega_C) \to \mathcal{A}[\mathcal{L}_C(d)], \quad \omega \mapsto \big(g_1(F_1), \ldots, g_d(F_d)\big).$$

Hénaut [2] has checked that this map is injective (the main idea is that if all g_i vanish, then ω vanishes on each branch, hence globally), so

$$\dim_{\mathbb{C}} \mathcal{A}[\mathcal{L}_C(d)] \geq \dim_{\mathbb{C}} H^0(C, \omega_C) = \frac{(d-1)(d-2)}{2}.$$

By Bol's inequality (Theorem 1.6), $\mathrm{rk}\mathcal{L}_C(d) \leq \frac{(d-1)(d-2)}{2}$, so equality holds and $\mathcal{L}_C(d)$ has maximal rank. Thus, we obtain the following.

Proposition 2.2. *Every algebraic planar d-web is of maximal rank and attains Bol's bound in Theorem 1.6.*

In other words, Abel's theorem explains how Abelian differentials on a plane curve produce Abelian relations for the associated web, and why algebraic webs realize the extremal rank.

2.3 Lie-Darboux-Griffiths theorem

Definition 2.3. Let $\mathcal{W} = \mathcal{W}(F_1, \ldots, F_d)$ be a planar d-web. An Abelian relation $\sum_{i=1}^d g_i(F_i)\mathrm{d}F_i = 0$, with $g_i \in \mathbb{C}\{z\}$ as in Definition 1.5, is *algebraizing* if none of the 1-forms $g_i(F_i)\mathrm{d}F_i$ vanishes identically. Equivalently, every foliation of the web appears nontrivially in the relation [2].

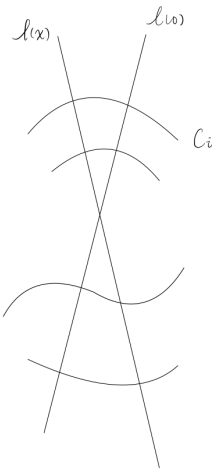


Figure 5: Local curve germs C_i meet $l(0)$ transversely, and $p_i(x) \in C_i \cap l(x)$ for a line $l(x)$ close to $l(0)$. This is the local setup behind algebraizing relations in the Lie-Darboux-Griffiths discussion [4].

The next theorem, which can be viewed as a converse to Abel's theorem, partly answers the problem of determining which webs $\mathcal{W}(d)$ on $(\mathbb{C}^2, 0)$ have maximal rank and are algebraizable.

Theorem 2.4 (Lie-Darboux-Griffiths)**.** *Let $\mathcal{L}(d)$ be a linear d-web on $(\mathbb{C}^2, 0)$ with $d > 3$. If $\mathcal{L}(d)$ admits at least one algebraizing Abelian relation, then $\mathcal{L}(d)$ is algebraic: there exists a reduced algebraic curve $C \subset \mathbb{P}^2$ of degree d, not necessarily irreducible and possibly singular, such that $\mathcal{L}(d)$ is equivalent to $\mathcal{L}_C(d)$.*

In particular, any linearizable planar d-web of maximal rank is algebraizable [2].

Together, Proposition 2.2 and Theorem 2.4 give a beautiful picture we are always hoping for (dreaming of). Algebraic curves yield maximal-rank linear webs via Abel's theorem, and conversely, a linear web with an algebraizing Abelian relation must arise from such a curve.

3 3-webs

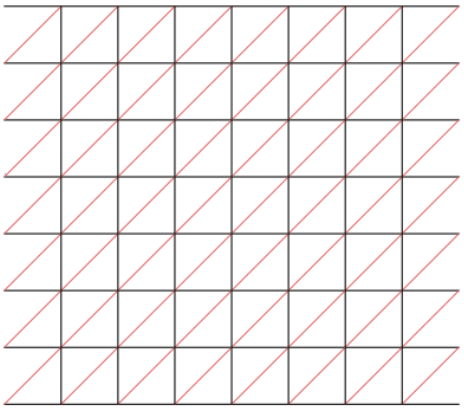
Although this section is more or less distinct from the other sections, 3-webs have some really interesting features that I wanted to discuss. There is another construction due to W. Blaschke.

Let $\mathcal{W} = \mathcal{W}(\omega_1, \omega_2, \omega_3)$ be a planar 3-web, given in terms of Pfaffian 1-forms ω_i defining the foliations. Since ω and $\rho\omega$ define the same foliation for any nowhere vanishing holomorphic function $\rho \in \mathcal{O}^*$, we may (and do) assume that $\omega_1 + \omega_2 + \omega_3 = 0$.

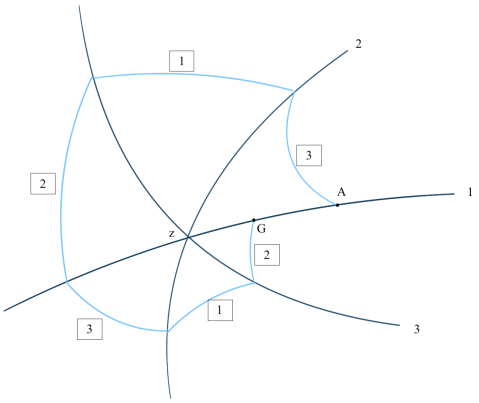
Set $\Omega := \omega_1 \wedge \omega_2$. Then $\Omega = \omega_2 \wedge \omega_3 = \omega_3 \wedge \omega_1$ is a nowhere vanishing 2-form. There is a unique 1-form γ such that $\mathrm{d}\omega_i = \gamma \wedge \omega_i$ for $i = 1, 2, 3$, and the 2-form $K(\mathcal{W}) := \mathrm{d}\gamma$ is the *Blaschke curvature* of the web [2].

Definition 3.1. A planar 3-web is *parallelizable* if it is equivalent to $\mathcal{W}(x, y, x + y)$, i.e. to the web whose three foliations are the straight lines $x = \text{const}$, $y = \text{const}$, and $x + y = \text{const}$.

It is *hexagonal* if the small hexagons obtained by successively following the three foliations around a point always close; see [2] for the construction.



(a) A parallelizable 3-web (locally equivalent to $\mathcal{W}(x, y, x + y)$).



(b) Hexagon closure construction for a 3-web [2].

The classical result of Thomsen, Blaschke, and Chern [2] can be phrased as follows.

Theorem 3.2. *For a planar 3-web $\mathcal{W}$, the following are equivalent:*

(i) $\mathrm{rk}\mathcal{W} = 1$ (*maximal rank*);

(ii) *$\mathcal{W}$ is parallelizable;*

(iii) *$\mathcal{W}$ is hexagonal;*

(iv) *the Blaschke curvature vanishes: $K(\mathcal{W}) = 0$.*

Example 3.3 (cf. Hénaut [2])**.** Suppose the foliations are given by $x = \text{const}$, $y = \text{const}$, and $F(x, y) = \text{const}$, and work on the open set where $F_x F_y \neq 0$. We choose Pfaffian 1-forms for these foliations, normalized by $\omega_1 = F_x \, \mathrm{d}x$, $\omega_2 = F_y \, \mathrm{d}y$, and $\omega_3 = -\mathrm{d}F = -F_x \mathrm{d}x - F_y \mathrm{d}y$, so that $\omega_1 + \omega_2 + \omega_3 = 0$. We compute

$$K(\mathcal{W}_F) = \partial_x \partial_y \left(\log \frac{F_x}{F_y} \right) \mathrm{d}x \wedge \mathrm{d}y.$$

Example 3.4 (Logarithmic hexagonal web cf. Hénaut [2])**.** On any simply connected open set where branches of the logarithm are chosen, we have the functional relation $\log x - \log y + \log \frac{y}{x} = 0$. Differentiating gives $\frac{\mathrm{d}x}{x} - \frac{\mathrm{d}y}{y} + \mathrm{d} \log \frac{y}{x} = 0$, so we obtain an Abelian relation for the 3-web $\mathcal{W}(x, y, y/x)$. Outside its singular locus $S = \{xy = 0\}$, this web is hexagonal and hence parallelizable. It is the web formed by the three pencils of lines in $\mathbb{P}^2$ with vertices $[0, 0, 1]$, $[0, 1, 0]$, and $[1, 0, 0]$.

Remark 3.5. Although Definition 1.1 is stated for holomorphic germs $F_i \in \mathbb{C}\{x, y\}$ on $(\mathbb{C}^2, 0)$, many examples are written using rational functions (e.g. y/x, $(1 - y)/(1 - x)$) or functions such as $\log$ and Li_2. In such cases, we choose a basepoint $p \in \mathbb{C}^2$ away from the singular locus of the defining functions, and we assume the transversality condition $\mathrm{d}U_i(p) \wedge \mathrm{d}U_j(p) \neq 0$ for $i \neq j$. After translating p to the origin and replacing each first integral U_i by $U_i - U_i(p)$, we obtain holomorphic germs vanishing at 0 that define the same foliations. Then we are good.

4 Linearization of planar webs

4.1 Linearizable webs

Recall that a web is *linearizable* if it is equivalent to a linear web. Poincaré showed that maximal-rank 4-webs are automatically linearizable. In this section, we briefly go through the key argument.

Proposition 4.1 (Poincaré)**.** *Let $\mathcal{W}$ be a planar 4-web with $\mathrm{rk}\mathcal{W} = 3$ (maximal). Then $\mathcal{W}$ is linearizable* [2].

Combining Proposition 4.1 with Theorem 2.4, we obtain the following.

Corollary 4.2. *Every planar 4-web of maximal rank is algebraizable: it is equivalent to the algebraic web $\mathcal{L}_C(4)$ associated to some reduced plane quartic $C \subset \mathbb{P}^2$.*

4.2 Hénaut's polynomial $P_{\mathcal{W}}$

After a linear change of coordinates, the foliations of a planar d-web $\mathcal{W}(d)$ on $(\mathbb{C}^2, 0)$ can be written as $X_i = \partial_x + b_i(x,y)\partial_y$, $1 \le i \le d$, with holomorphic functions $b_i \in \mathcal{O}$ satisfying $b_i(0) \neq b_j(0)$ for $i \neq j$. Hénaut [2] associates to $\mathcal{W}$ a unique polynomial

$$P_{\mathcal{W}}(x,y,p) = P_0(x,y) + P_1(x,y)p + \cdots + P_{d-1}(x,y)p^{d-1} \in \mathcal{O}[p]$$

characterized by

$$P_{\mathcal{W}}(x,y,b_i(x,y)) = X_i\big(b_i(x,y)\big) \quad \text{for } 1 \le i \le d.$$

With this normalization, each leaf of the web is the graph of a function $y(x) \in \mathbb{C}\{x\}$ that is an integral curve of the second-order differential equation $y'' = P_{\mathcal{W}}\big(x, y, y'\big)$. He then proves the following.

Theorem 4.3 (Hénaut)**.** *Let $\mathcal{W}$ be a planar d-web with associated polynomial $P_{\mathcal{W}}(x,y,p)$ characterized by $P_{\mathcal{W}}(x,y,b_i) = X_i(b_i)$. Then $\mathcal{W}$ is linearizable if and only if the differential system admits an analytic solution $(P_0, P_1, P_2, P_3) \in \mathcal{O}^4$.*

For $d > 4$, this is equivalent to requiring that

$$P_{\mathcal{W}}(x,y,p) = P_0(x,y) + P_1(x,y)p + P_2(x,y)p^2 + P_3(x,y)p^3,$$

i.e. $\deg_p P_{\mathcal{W}} \le 3$*, and that (P_0, P_1, P_2, P_3) satisfies the integrability system (S). In that case, the linearization is unique up to projective transformations* [2].

Remark 4.4. Hénaut also gives a connection-theoretic reformulation of this theorem in [3].

We will use this in Section 6 to see that Bol's 5-web is not linearizable (for that web, $\deg_p P_B = 4$) [2].

5 Pirio's ODE method

Our eventual goal is to understand the construction and interpretation of *exceptional* planar 5-webs in the sense of Definition 1.8.

Pirio develops a differential equation approach to Abelian relations [1]. For some normal forms, this yields explicit differential constraints that capture maximal rank. In Section 6 we will apply this technique, together with Hénaut's linearization criterion, to exhibit exceptional planar 5-webs.

Given a planar d-web $\mathcal{W} = \mathcal{W}(U_1, \dots, U_d)$, consider the functional equation

$$\sum_{i=1}^{d} F_i\big(U_i(x,y)\big) = \text{const} \tag{2}$$

in one variable holomorphic germs F_i. Any solution $(F_1, \dots, F_d)$ yields an Abelian relation in the sense of Definition 1.5 by differentiating (2), and conversely, local Abelian relations integrate to solutions of (2).

5.1 Operator $\mathcal{P}_{\mathcal{W}}$

Pirio shows that, starting from the functional equation (2), we can eliminate $F_1, \dots, F_{d-1}$ and their derivatives to obtain a scalar linear differential operator

$$\mathcal{P}_{\mathcal{W}} \in \mathcal{M}_1[\partial_z],$$

where $\mathcal{M}_1$ denotes the field of one-variable meromorphic germs, and the coefficients of $\mathcal{P}_{\mathcal{W}}$ are nonlinear differential expressions in the defining first integrals U_i. For every Abelian relation $(F_1, \dots, F_d)$ of $\mathcal{W}$, the last component F_d satisfies $\mathcal{P}_{\mathcal{W}}(F_d) = 0$. Equivalently, if π_d denotes the projection of $\mathcal{A}[\mathcal{W}]$ onto the last component $\in \mathbb{C}\{z\}/\mathbb{C}$, then $\pi_d(\mathcal{A}[\mathcal{W}]) \subset \mathrm{Sol}(\mathcal{P}_{\mathcal{W}})$ [1]. In particular,

$$\dim_{\mathbb{C}} \pi_d(\mathcal{A}[\mathcal{W}]) \leq \mathrm{Ord}(\mathcal{P}_{\mathcal{W}}) - 1.$$

Moreover, the rank decomposes as $\mathrm{rk}\mathcal{W} = \mathrm{rk}\mathcal{W}(U_1, \dots, U_{d-1}) + \dim_{\mathbb{C}} \pi_d\big(\mathcal{A}[\mathcal{W}]\big)$, and combining this with Bol's inequality yields the necessary condition

$$\mathrm{Ord}(\mathcal{P}_{\mathcal{W}}) - 1 \geq \frac{(d-1)(d-2)}{2} - \mathrm{rk}\mathcal{W}(U_1, \dots, U_{d-1}),$$

which translates into explicit differential constraints on the U_i [1].

5.2 Webs with a parallelizable 4-subweb

Pirio then discusses 5-webs containing a parallelizable 4-subweb. This is his most interesting result. Up to equivalence, any such web can be written as

$$\mathcal{W}_{a,U} = \mathcal{W}(x, y, x+y, y+ax, U(x,y)), \qquad a \in \mathbb{C} \setminus \{0,1\},$$

where the first four foliations are linear and $U \in \mathcal{O}$ is such that the discriminant

$$\Delta := U_x U_y (U_x - U_y)(U_x - aU_y)$$

does not vanish at the origin [1].

Writing $\partial_U := U_y \partial_x - U_x \partial_y$ for the vector field tangent to the leaves of the fifth foliation, Pirio proved the following.

Theorem 5.1 (Pirio). *There exist explicit rational functions $\Omega_1, \Omega_2, \Omega_3$ and a discriminant $\Delta = U_x U_y (U_x - U_y)(U_x - aU_y)$ such that for the web $\mathcal{W}_{a,U}$ the following are equivalent:*

(i) *$\mathcal{W}_{a,U}$ has maximal rank 6;*

(ii) *for $i = 1, 2, 3$, the function Ω_i/Δ is constant along the leaves of the fifth foliation, i.e. $\partial_U \left(\frac{\Omega_i}{\Delta}\right) = 0$.*

Equivalently, each Ω_i/Δ is a one variable function of U [1].

This gives an explicit system of PDEs whose solutions yield candidate maximal-rank 5-webs. In Section 6 we will present Pirio's exceptional examples.

6 Exceptional planar 5-webs and polylogarithms

6.1 Exceptional webs

Recall that in the planar case ($n = 1$), we call a d-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$ *exceptional* if it has maximal rank $\mathrm{rk}\mathcal{W} = \frac{(d-1)(d-2)}{2}$ and is not algebraizable (Definition 1.8).

More generally, for webs $\mathcal{W}(d, 2, n)$ of codimension n on $(\mathbb{C}^{2n}, 0)$ there is an n-rank $\mathrm{rk}_n\mathcal{W}$, defined in terms of Abelian relations of degree n. Chern and Griffiths proved the sharp inequality

$$\mathrm{rk}_n\mathcal{W} \leq \binom{d-1}{n+1},$$

with equality for algebraic webs coming from projective hypersurfaces in $\mathbb{P}^{n+1}$ [2].

Following [2], a germ of d-web $\mathcal{E}(d, 2, n)$ of codimension n in $(\mathbb{C}^{2n}, 0)$ is called *exceptional* if

(i) it has maximal n-rank: $\mathrm{rk}_n\mathcal{E} = \binom{d-1}{n+1}$;

(ii) it is not algebraizable (i.e. not equivalent to any algebraic web coming from a projective hypersurface).

For $n = 1$ this recovers Definition 1.8. In particular, from our results on planar webs of degree $d \leq 4$ (Theorem 1.6, Proposition 2.2, and the linearization/algebraization results for 3- and 4-webs), exceptional planar webs, if they exist, must have $d \geq 5$. Bol's classical 5-web is the basic example. We devote the rest of this section to Bol's web and to Pirio's new exceptional 5-webs.

6.2 Bol's exceptional 5-web and logarithmic relations

Bol's classical web is the planar 5-web

$$B = \mathcal{W}\left(x, y, \frac{x}{y}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right).$$

It has $\mathrm{rk}B = 6 = \frac{(5-1)(5-2)}{2}$, so B has maximal rank but, by Theorem 4.3, is not linearizable (Hénaut computed that $\deg_p P_B = 4$); hence B is not algebraizable and is an exceptional planar web [2].

Hénaut emphasizes that the Abelian relations of B are described by functional equations for the logarithm and dilogarithm [2].

Example 6.1 (Logarithmic relations)**.** The logarithm $\phi_1(z) = \log z$ satisfies $\phi_1(x) - \phi_1(y) + \phi_1\left(\frac{y}{x}\right) = 0$, for suitable choices of branches on a simply connected domain avoiding $\{xy = 0\}$. Differentiating gives $\frac{\mathrm{d}x}{x} - \frac{\mathrm{d}y}{y} + \mathrm{d}\log\frac{y}{x} = 0$, which is an Abelian relation for the 3-subweb $\mathcal{W}(x, y, y/x) \subset B$ (Example 3.4). Similar relations for other triples of arguments give five linearly independent logarithmic Abelian relations for B.

6.3 The Rogers dilogarithm

Motivated by a discussion with Professor Kerr, we now explain how the classical five-term functional equation for the *Rogers dilogarithm* produces the sixth (essentially dilogarithmic) Abelian relation for Bol's web.

Following Zagier [5], the Rogers dilogarithm is defined on $0 < x < 1$ by

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x),$$

and admits analytic continuation once we choose branches of $\log(z)$ and $\log(1-z)$. Locally, we can fix a simply connected open set in the (x,y)-plane avoiding the singular locus of the rational functions, so that the branches of $\log$, and hence of L, are holomorphic.

Zagier notes that (up to additive constants and mod $\pi^2/2$ on $\mathbb{P}^1(\mathbb{R})$) L satisfies the reflection and inversion identities $L(z)+L(1-z)=$ const and $L(z)+L(1/z)=$ const. Thus, on a simply connected domain with fixed branches, we have $\mathrm{d}L(1-z)=-\mathrm{d}L(z)$ and $\mathrm{d}L(1/z)=-\mathrm{d}L(z)$.

Moreover, L satisfies the *five-term functional equation*

$$L(u)+L(v)+L\left(\frac{1-u}{1-uv}\right)+L(1-uv)+L\left(\frac{1-v}{1-uv}\right)=\text{const}, \tag{3}$$

for (u,v) in a domain where all five arguments avoid $\{0,1,\infty\}$ [5]. Differentiating (3) eliminates the additive-constant ambiguity and yields an identity of holomorphic 1-forms:

$$\mathrm{d}L(u)+\mathrm{d}L(v)+\mathrm{d}L\left(\frac{1-u}{1-uv}\right)+\mathrm{d}L(1-uv)+\mathrm{d}L\left(\frac{1-v}{1-uv}\right)=0. \tag{4}$$

Now make the substitution $u=\frac{y}{x}$ and $v=\frac{1-x}{1-y}$ so that $\frac{1-u}{1-uv}=1-y$, $\frac{1-v}{1-uv}=x$, and $uv=\frac{y(1-x)}{x(1-y)}$. Set $w:=\frac{x(1-y)}{y(1-x)}$ so that $1-uv=1-\frac{1}{w}$. Substituting into (4) yields

$$\mathrm{d}L\left(\frac{y}{x}\right)+\mathrm{d}L\left(\frac{1-x}{1-y}\right)+\mathrm{d}L(1-y)+\mathrm{d}L\left(1-\frac{1}{w}\right)+\mathrm{d}L(x)=0.$$

Using $\mathrm{d}L(1-y)=-\mathrm{d}L(y)$ and $\mathrm{d}L(1-1/w)=-\mathrm{d}L(1/w)=\mathrm{d}L(w)$, we obtain the equivalent relation

$$\mathrm{d}L(x)-\mathrm{d}L(y)+\mathrm{d}L\left(\frac{y}{x}\right)+\mathrm{d}L\left(\frac{1-x}{1-y}\right)+\mathrm{d}L\left(\frac{x(1-y)}{y(1-x)}\right)=0. \tag{5}$$

This is a holomorphic Abelian relation for the 5-web $\mathcal{W}\left(x,y,\frac{y}{x},\frac{1-x}{1-y},\frac{x(1-y)}{y(1-x)}\right)$ on any simply connected domain where the five arguments avoid $\{0,1,\infty\}$ and where branches have been fixed. Since the foliations defined by y/x and x/y (resp. by $(1-x)/(1-y)$ and $(1-y)/(1-x)$) have the same leaves, this web is equivalent to Bol's web.

Together with the five logarithmic Abelian relations discussed above, the dilogarithmic relation (5) yields the full 6-dimensional space $\mathcal{A}[B]$ of Abelian relations, in agreement with Hénaut's description of $V_{\log}(5)=B$ [2].

6.4 Pirio's exceptional 5-webs

Pirio applies Theorem 5.1 to construct new planar 5-webs of maximal rank that are not algebraizable. Any germ of a planar 5-web admitting a parallelizable 4-subweb is equivalent to a web of the form $\mathcal{W}_{a,U}$ from Section 5.

Now consider $a=-1$ and assume $U(x,y)=A(x)+B(y)$. Then the PDE constraints in Theorem 5.1 can be solved explicitly. This gives new exceptional 5-webs:

$$\begin{aligned} W_1&=\mathcal{W}(x,y,x+y,x-y,x^2+y^2),\\ W_2&=\mathcal{W}(x,y,x+y,x-y,e^x+e^y),\\ W_3&=\mathcal{W}(x,y,x+y,x-y,xy). \end{aligned}$$

Pirio shows that each of these has rank 6 (maximal) but is not algebraic, and that no two of them are equivalent, nor is any of them equivalent to Bol's web B [1]. Since any linearizable planar

5-web of maximal rank is algebraizable (Theorem 2.4), these webs are also not linearizable. Thus W_1, W_2, W_3 are three pairwise inequivalent exceptional 5-webs, answering Chern's problem about the uniqueness of Bol's example [1].

6.5 Polylogarithmic webs

Hénaut introduces *polylogarithmic webs*, which are obtained from functional equations of the polylogarithms $\phi_p(z)$ of order $p \geq 1$.

The main idea is that functional identities of the form

$$\sum_i c_i \, \phi_p(U_i(x,y)) + (\text{lower polylogs}) = \text{const},$$

with rational functions U_i, can be differentiated to an Abelian relation $\sum_i G_i\big(U_i(x,y)\big)\mathrm{d}U_i = 0$ for the web generated by the U_i, where the G_i are one-variable combinations of ϕ_p and lower-order polylogarithms [2].

We have already met:

- $p = 1$: the 3-term relation for $\log$ yields the hexagonal log web of Example 3.4.
- $p = 2$: the 5-term relation for the dilogarithm yields Bol's web B.

For the trilogarithm ϕ_3, Hénaut showed that there is a 9-term functional relation whose differential defines a 9-web $V_{\log}(9)$ on $\mathbb{C}^2$ containing Bol's web B as a subweb. This web cannot be linearized, and is therefore a candidate for another exceptional planar web [2].

These polylogarithmic examples illustrate an interesting phenomenon we already saw earlier in Abel's work. Functional equations for special functions (logarithm, dilogarithm, trilogarithm, ...) translate, via differentiation, into Abelian relations of planar webs. They sit at the intersection of web geometry, algebraic geometry, and the arithmetic of polylogarithms.

References

[1] L. Pirio, On maximal rank planar webs and Chern's problem, *C. R. Acad. Sci. Paris, Ser. I* **339** (2004), no. 2, 131–136.

[2] A. Hénaut, Analytic web geometry, in *Web Theory and Related Topics* (Toulouse, 1996), J. Grifone and É. Salem (eds.), World Sci. Publ., River Edge, NJ, 2001, pp. 6–47.

[3] A. Hénaut, On planar web geometry through abelian relations and connections, *Ann. of Math.* (2) **159** (2004), no. 1, 425–445.

[4] P. A. Griffiths, On Abel's differential equations, in *Algebraic Geometry: The Johns Hopkins Centennial Lectures*, J.-I. Igusa (ed.), Johns Hopkins Univ. Press, Baltimore, 1977, pp. 26–51.

[5] D. Zagier, The dilogarithm function, in *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization*, P. Cartier, P. Moussa, B. Julia, and P. Vanhove (eds.), Springer, Berlin–Heidelberg, 2007, pp. 3–65.

[6] M. Kerr, Abel's Theorem, Part I, Chapter 30 in *An Invitation to Complex Algebraic Geometry*, unpublished manuscript, 2025.

[7] M. Kerr, Abel's Theorem, Part II, Chapter 31 in *An Invitation to Complex Algebraic Geometry*, unpublished manuscript, 2025.