Spring 5-6-2026

A Generalization of Molino's Theory to Riemannian Groupoids

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A Generalization of Molino's Theory to Riemannian Groupoids

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A thesis presented to
Washington University in St. Louis
in partial fulfillment of the
requirements for the degree
of Master of Arts

May 2026
St. Louis, Missouri

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List of Symbols

$\mathcal{F}$, $T(\mathcal{F})$, L_x	A regular foliation; $T(\mathcal{F}) \subseteq TM$ is the involutive subbundle tangent to the foliation, and L_x is the leaf through x.
q, $N(\mathcal{F}) := TM/T(\mathcal{F})$	The codimension of the foliation $\mathcal{F}$ and its normal bundle.
$G \rightrightarrows M$	A Lie groupoid with object manifold M and arrow manifold G.
s, t, u, i, m	Source, target, unit, inversion, and multiplication maps of a Lie groupoid.
$G^{(n)}$	The manifold consisting of chains of n composable arrows; $G^{(0)} = M$, $G^{(1)} = G$, and $G^{(2)} = G \times_M G$. On $G^{(2)}$, $\pi_1(h, g) = h$, $\pi_2(h, g) = g$, and $m(h, g) = hg$.
G_x, O_x, $G\|_O$	The isotropy group $G_x := \{g \in G \mid s(g) = t(g) = x\}$, the orbit $O_x := t(s^{-1}(x))$, and the restriction groupoid to an orbit $O \subseteq M$.
$A_G := \mathrm{Lie}(G)$, ρ_G	The Lie algebroid attached to $G \rightrightarrows M$, with $(A_G)_x = \ker(ds)_{1_x}$ and anchor $\rho_G = dt\|_{A_G} : A_G \to TM$.
λ_g, λ^N, λ^N_x	Normal representation: $\lambda_g([v]) := [dt_g(X)]$, where $ds_g(X) = v$; in the regular case the orbitwise maps assemble to $\lambda^N : G \times_M N \to N$, with isotropy representation $\lambda^N_x : G_x \to O(N_x, g^N_x)$.
$\eta^{(2)}$, $\eta^{(1)}$, $\eta^{(0)}$	A 2-metric on $G^{(2)}$, the induced 1-metric on $G^{(1)}$, and the induced 0-metric on $G^{(0)} = M$.
$\mathcal{O}$, $N := TM/T(\mathcal{O})$, q	The orbit foliation of a regular groupoid; N is its normal bundle and $\operatorname{rank} N = q = \operatorname{codim} \mathcal{O}$.
g^M, g^N	The induced metric $g^M := \eta^{(0)}$ on M and the induced metric g^N on N.
$OF(M, \mathcal{O})$, π	The transverse orthonormal frame bundle $\pi : OF(M, \mathcal{O}) \to M$.
$\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{O}}$	Lifted foliations on transverse orthonormal frame bundles.
θ, ω, ∇^{tr}	The transverse canonical form, transverse Levi–Civita connection form, and transverse Levi–Civita connection on N.
$\kappa : OF(M, \mathcal{O}) \to B$, L_b	The $O(q)$-equivariant fibre bundle whose fibres are the closures of the leaves of $\widetilde{\mathcal{O}}$; $L_b := \kappa^{-1}(b)$.
$L(M, \mathcal{F})$, $l(M, \mathcal{F})$	The Lie algebra of projectable vector fields and the quotient $l(M, \mathcal{F}) := L(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$.
$\mathcal{F}_{\mathrm{bas}}$, $\pi_{\mathrm{bas}} : X \to W$	The basic foliation and the basic fibre bundle $\pi_{\mathrm{bas}} : X \to W := X/\mathcal{F}_{\mathrm{bas}}$.

$A := b(X, \mathcal{F}) \to W$, ρ_A	The transitive basic Lie algebroid, with $\Gamma(A) \cong l(X, \mathcal{F})$ and anchor $\rho_A : A \to TW$.
$(\ker \rho_A)_b$	The isotropy Lie algebra of the basic Lie algebroid; in the regular-groupoid theorem, $(\ker \rho_A)_b \cong l(L_b, \widetilde{\mathcal{O}}\vert_{L_b})$.
ω^b_{MC}, $\omega^{\pi_{\mathrm{bas}}}_{\mathrm{MC}}$	The Maurer–Cartan form on a Molino fibre and the global section whose restriction to each fibre corresponds to that Maurer–Cartan form.
$\mathfrak{g}$	The structural Lie algebra of the fibrewise Lie foliations.
Q, $\mathcal{F}_Q$, $\bar{g}$	An orbifold, an orbifold foliation, and a transverse metric on $(Q, \mathcal{F}_Q)$.
$\mathcal{O} := OF(Q, \mathcal{F}_Q)$, $\widetilde{\mathcal{F}}_Q$, θ_Q, ω_Q	In the orbifold setting, the transverse orthonormal frame bundle, its lifted foliation, transverse canonical form, and transverse Levi–Civita connection form.
$\mathcal{N}$, $\bar{s}$, $\bar{s}_G$	The manifold $\mathcal{N}$ with a smooth right $O(q)$-action, the $O(q)$-equivariant orbifold fibre bundle $\bar{s} : \mathcal{O} \to \mathcal{N}$, and the corresponding bundle $\bar{s}_G : \mathcal{O}_G \to \mathcal{N}$.
$\mathcal{O}_G$, Φ	$\mathcal{O}_G := OF(M, \mathcal{F})/G$ is the orbit space of the lifted G-action; $\Phi : \mathcal{O}_G \xrightarrow{\cong} OF(Q, \mathcal{F}_Q)$ is the canonical $O(q)$-equivariant foliated orbifold isomorphism.
$\widetilde{\mathcal{F}}_G$, θ_G, ω_G	The descended foliation and descended orbifold 1-forms on $\mathcal{O}_G$, satisfying $\Phi^*\theta_Q = \theta_G$ and $\Phi^*\omega_Q = \omega_G$.

Acknowledgments

First and foremost, I want to express my deepest gratitude to my advisor, Professor Xiang Tang. Words cannot express how much I appreciate him for cultivating my interest in mathematical research. He is always patient, encouraging, and motivating in our weekly meetings. I want to thank him for all of his time and effort in training me to think more deeply about math, to reach for the foundations behind concepts and see how they build up into theories. What I am most grateful for is not only what he taught me, but the way he taught me to think. Although I have never told him this directly in person, without his support and inspiration, I never imagined I would be this ambitious about this project. He is not only an extremely talented mathematician, as everyone knows, but also a caring advisor who wants me to do well. He is my true benefactor. I would also like to thank him for bringing me to the most beautiful corner of mathematics: noncommutative geometry and Poisson geometry.

I would also like to thank Professor Yanli Song for the directed reading in K-theory and index theory, and for his topics course on symplectic geometry and geometric quantization in Fall 2025, where I encountered many applications of equivariance, group actions, tubular neighborhoods, and much else—all of which found their way into this thesis. I thank Professor Aliakbar Daemi for agreeing to serve on my thesis committee. I thank Professor Matt Kerr for teaching me algebraic geometry and for a project on foliations in web geometry. As an extension of this project, the desingularization results lean on techniques I first picked up in his office. I hope that, if time allows, the results in this thesis can extend in some way into what they taught me.

I also want to share a personal story that will probably remain one of the best memories of my life. When I was still a kid back in the day in Canada, I loved skiing fast and wanted to become a professional skier; in college I finally had the chance to train full time. I want to thank all my coaches. Tony used to say, “The snow doesn’t ski itself.” Dave would ask, “What’s the worst that can happen?” when I was scared of the speed. These things may seem irrelevant to other mathematicians, but they taught me courage, discipline, and how to accept consequences. In November 2025, on a chairlift at Copper Mountain, Ashley and I talked about creativity—life isn’t black and white, and nothing goes linearly or commutatively. Skiing, I still think, is the greatest sport in the universe: a dance between gravity and space, a freedom painted in white, a silent symphony of speed, solitude, and soul.

My advisor may be finding out for the first time (haha!) that I was still racing FIS during my first year at WashU. I once drove almost twenty hours back from a race in Canada for a complex analysis midterm; I was trying to pick up the lectures from the very beginning of the class notes during gas stops and McDonald’s visits. I didn’t crash in the race, but I didn’t recover the lectures either. This is probably a striking example of noncommutativity: had I studied for the exam first and skied afterwards, the result would almost certainly have been different. My weekly travels to ski in Minnesota, Colorado, and Utah were some of my best memories of first-year graduate school. I do wonder, occasionally, whether I would be better at math without the three concussions I picked up along the way. I have always wanted to say that my life is not only mathematics—and yet, without mathematics, would my life still be my life?

I also want to thank the friends who made math feel like the air we breathe. And of course the ones who golf, ski, boat, and drift with me on mountain roads and race tracks; the ones who talk car mods and mechanics; the ones who work through the restaurant checklist with me around the world; and the ones who discuss gardening, plumbing, interior design, and handyman work. The list goes on. Thank you all for making my life interesting in a nontrivial way. I suspect I won't love doing this many things forever, so it felt worth writing down while I still do.

None of this would have been possible without my parents. My father once dreamed of becoming a mathematician, and for as long as I can remember has hoped I would be one. I'm not there yet, but I guess I will be soon. My mother, who always claimed math was her least favorite subject, has loved me unconditionally and been proud of me through everything. I am also lucky to belong to a generation that lets me simply think about mathematics and live however I like. I love them from the bottom of my heart.

Life is my playground.

Lily Zhang

Washington University in Saint Louis

May 2026

To my dearest mountains.

致我最敬爱的大山。

ABSTRACT OF THE THESIS

A Generalization of Molino's Theory to Riemannian Groupoids

by

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Master of Arts in Mathematics

Washington University in St. Louis, 2026

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Riemannian groupoids describe Riemannian foliations together with their symmetries. In this thesis, we extend classical Molino's theory, which concerns the structure of Riemannian foliations on compact manifolds, to the setting of regular Riemannian groupoids with compact unit spaces.

We observe that the orbit foliation associated with a regular Riemannian groupoid defines a Riemannian foliation on the unit space. The main result of the thesis shows that the normal representation of a Riemannian groupoid extends naturally to an action on the Molino structures of the orbit foliation, thereby yielding the fundamental structural description of a regular Riemannian groupoid. In addition to the main result, we clarify the essential role played by basic Lie algebroids in Molino's theory.

As applications, we derive Molino-type structure theorems for compact group actions and orbifolds.

Chapter 1

Introduction

Molino's structure theorem did not arise in isolation. Molino introduced it as part of a line of development beginning with the global study of ordinary differential equations, where a nonsingular vector field partitions a manifold into its trajectories, and continuing through the general formulation of foliations by Ehresmann and Reeb. Within this broader theory, Reinhart's 1959 introduction of bundle-like metrics singled out Riemannian foliations as those foliations whose leaves remain locally at constant distance from one another. Reinhart also established two of the rigidity phenomena that made the subject geometrically compelling: geodesics initially perpendicular to a leaf remain perpendicular to every leaf they meet, and for complete bundle-like metrics, the leaves have the same universal covering [13, Introduction].

Molino's insight was that the Riemannian condition is fundamentally transverse. In his original presentation, the global geometry becomes accessible only after passing from $(M, \mathcal{F})$ to the orthonormal transverse frame bundle and the lifted foliation, where the transverse Levi–Civita connection produces a canonical transverse parallelism. This reduces the study of compact Riemannian foliations to the transversely parallelizable and Lie foliation settings, and leads to the description of leaf closures in terms of a locally constant sheaf of germs of transverse Killing fields whose orbits are exactly the leaf closures [13, Introduction, Thm. 5.2].

In recent years, Molino's theory has been pushed in several directions. Liu [10] proposed a generalization involving a Riemannian foliation with a foliated projection onto a second one, yielding, in the Killing case, an equivariant basic cohomology isomorphism—extending Goertsches–Töben—that is realized geometrically through equivariant basic $\widehat{A}$-genus characters and linked to the transverse Atiyah–Singer index theorem. Álvarez López and Moreira Galicia [1] developed a topological analogue for compact equicontinuous foliated spaces with dense leaves, associating to such a space a structural local group. Dyer, Hurder, and Lukina [5] developed a Molino theory for matchbox manifolds, which are foliated spaces with totally disconnected transversals.

In each of these developments, the starting object is still a foliation; in this thesis, we start instead from a Lie groupoid and place it in a Riemannian setting by means of a 2-metric. More precisely, the main theorem is proved for Hausdorff regular Lie groupoids with a compact connected object manifold, equipped with such a metric. When a Lie groupoid is regular, the connected components of its orbits form the orbit foliation [11], so Molino's theory provides the natural regular model. The point of the groupoid viewpoint, however, is that it retains structure that is invisible from the orbit foliation alone, for example, isotropy and the transverse normal representation. The thesis shows how Molino's theory for the orbit foliation can be recovered in this setting and then reinterpreted in groupoid terms.

1.1 Main results

The main theorem for the regular Riemannian groupoid case is Theorem 3.41. Let $G \rightrightarrows M$ be a Hausdorff regular Lie groupoid equipped with a 2-metric, with M compact and connected.

Then the induced orbit foliation $(M, \mathcal{O})$ is Riemannian. Passing to the transverse orthonormal frame bundle $OF(M, \mathcal{O})$ and to the lifted foliation $\widetilde{\mathcal{O}}$, we recover the Molino package.

The Lie algebroid refinement is that, on the foliation side, Proposition 3.37 constructs the transitive basic Lie algebroid and Lemma 3.38 identifies its isotropy fibres with the structural Lie algebras of the Lie foliations. Proposition 3.49 constructs a global Maurer–Cartan form assembling the fibrewise Maurer–Cartan forms. On the groupoid side, Proposition 3.51 identifies $\Gamma(A_G)$ with right-invariant source-vertical vector fields on G. Proposition 3.55 shows that, in the regular case, its anchor recovers the orbit distribution and its isotropy recovers the isotropy Lie algebras, and Proposition 3.57 shows that the transverse isotropy representation differentiates to a Lie algebra representation of the isotropy Lie algebra on the transverse directions.

Chapter 4 also develops several corollaries of Molino's theory. Proposition 4.7 gives a compact-group-equivariant version of Molino's construction. Corollaries 4.10 and 4.38 then formulate corresponding orbifold and proper effective étale versions through the orbifold transverse frame-bundle construction.

1.2 Organization of the thesis

We begin in Chapter 2 by reviewing the background and basic terminology for foliations, Lie groupoids, Lie algebroids, and Riemannian groupoids. We develop the main regular theory in Chapter 3. We prove Molino's structure theorem for regular Riemannian groupoids and develop the basic Lie algebroid and the Lie algebroid $A_G = \mathrm{Lie}(G)$. Chapter 4 derives corollaries of Molino's theory in the compact-group-equivariant, orbifold, and proper effective étale groupoid cases. We end in Chapter 5 by illustrating the regular theory through the kernel groupoid and the Kronecker flow groupoid.

Chapter 2

Preliminaries

Our smooth-manifold terminology comes from Lee [6], and our Riemannian terminology comes from Lee [7].

For foliations and Lie groupoids, we follow Moerdijk–Mrčun [12]; for regular Lie groupoids, we also use Moerdijk's description of regular groupoids as extensions of foliation groupoids [11]. The Riemannian groupoid background comes from del Hoyo–Fernandes [4].

2.1 Foliations and transverse geometry

2.1.1 Regular foliations and basic forms

Definition 2.1 ([12, Secs. 1.1–1.2])**.** A *(regular) foliation* $\mathcal{F}$ of dimension p on a manifold M is an involutive rank-p subbundle $T(\mathcal{F}) \subseteq TM$. Its maximal connected immersed integral manifolds are the *leaves* of $\mathcal{F}$. The codimension of $\mathcal{F}$ is

$$q := \dim(M) - p,$$

and its normal bundle is

$$N(\mathcal{F}) := TM/T(\mathcal{F}).$$

We write L_x for the leaf through $x \in M$. A subset $U \subseteq M$ is *saturated* if it is a union of leaves. In a foliated chart $U \cong \mathbb{R}^p \times \mathbb{R}^q$, the leaves are locally the plaques $\mathbb{R}^p \times \{y\}$ [12, Sec. 1.1].

Definition 2.2 ([13, Sec. 2.3])**.** A differential form $\omega \in \Omega^k(M)$ is *basic* with respect to $\mathcal{F}$ if

$$\iota_X \omega = 0, \qquad \iota_X(d\omega) = 0$$

for every vector field $X \in \Gamma(T\mathcal{F})$.

Basic forms will reappear later in the descent arguments for transverse and orbifold forms.

2.1.2 Riemannian foliations and Molino's theorem

Definition 2.3 ([12, Sec. 2.2])**.** A Riemannian metric g on M is *bundle-like* for a foliation $\mathcal{F}$ if the induced metric on the normal bundle $N(\mathcal{F})$ is invariant under holonomy. A foliation is called *Riemannian* if it admits a bundle-like metric.

The term *bundle-like metric* goes back to Reinhart's 1959 paper [14].

For a compact connected Riemannian foliation $(\mathcal{F}, g)$ of codimension q on M, the transverse orthonormal frame bundle

$$\pi : OF(M, \mathcal{F}) \longrightarrow M$$

carries a lifted foliation $\widetilde{\mathcal{F}}$; moreover, the transverse canonical form and the transverse Levi–Civita connection form on $OF(M, \mathcal{F})$ furnish a transverse parallelism for $(OF(M, \mathcal{F}), \widetilde{\mathcal{F}})$ [12, Ex. 4.19 and Thm. 4.20].

Theorem 2.4 (Molino, [12, Thm. 4.26])**.** *Let $(\mathcal{F}, g)$ be a Riemannian foliation of a compact connected manifold M, and let $\tilde{\mathcal{F}}$ be the associated lifted foliation of the transverse orthonormal frame bundle $OF(M, \mathcal{F})$.*

1. *The foliated manifold $(OF(M, \mathcal{F}), \tilde{\mathcal{F}})$ is transversely parallelizable.*
2. *There exists a manifold N with an $O(q)$-action and an $O(q)$-equivariant fibre bundle $s : OF(M, \mathcal{F}) \to N$ such that the fibres of s are exactly the closures of the leaves of $\tilde{\mathcal{F}}$.*
3. *The Lie algebra $\mathfrak{g}$ of transverse vector fields of $\tilde{\mathcal{F}}|_{s^{-1}(y)}$ is independent, up to isomorphism, of $y \in N$. The foliation $\tilde{\mathcal{F}}|_{s^{-1}(y)}$ is a Lie foliation given by a canonical $\mathfrak{g}$-valued Maurer–Cartan form with a dense holonomy group.*

2.2 Lie groupoids, Lie algebroids, and actions

2.2.1 Lie groupoids, actions, orbits, and isotropy

Definition 2.5 ([12, Sec. 5.1])**.** A *Lie groupoid* $G \rightrightarrows M$ is a groupoid with object manifold M and arrow manifold G, where M is a smooth Hausdorff manifold and G is a smooth manifold, possibly non-Hausdorff, such that the source map

$$s : G \to M$$

is a smooth submersion with Hausdorff fibres, and all the other structure maps are smooth.

The other structure maps are the target map $t : G \to M$, the unit map $u : M \to G$, the inverse map $i : G \to G$, and the multiplication map

$$m : G \times_M G \to G, \qquad (h, g) \mapsto hg,$$

defined on the fibre product $G \times_M G := \{(h, g) \in G \times G \mid s(h) = t(g)\}$.

Definition 2.6 ([12, Sec. 5.3])**.** Let $G \rightrightarrows M$ be a Lie groupoid and let $\epsilon : N \to M$ be a smooth map. A *left action* of G on N along ϵ is a smooth map

$$G \times_M N := \{(g, y) \in G \times N \mid s(g) = \epsilon(y)\} \to N, \qquad (g, y) \mapsto g \cdot y,$$

such that

$$\epsilon(g \cdot y) = t(g), \qquad 1_{\epsilon(y)} \cdot y = y, \qquad g' \cdot (g \cdot y) = (g'g) \cdot y.$$

The associated *translation groupoid* (or *action groupoid*)

$$G \ltimes N \rightrightarrows N$$

has arrow manifold $G \times_M N$, source and target maps $s(g, y) = y$ and $t(g, y) = g \cdot y$, and multiplication $(g', g \cdot y)(g, y) = (g'g, y)$.

Taking $N = M$ and $\epsilon = \mathrm{id}_M$, any Lie groupoid acts canonically on its object manifold by $g \cdot x := t(g)$ whenever $s(g) = x$.

For $x \in M$, the *isotropy group* at x is

$$G_x := \{g \in G \mid s(g) = t(g) = x\},$$

and the *orbit* through x is

$$O_x := t\big(s^{-1}(x)\big) \subseteq M$$

[11, 1.5].

Definition 2.7 ([11, 1.3(b)]). A Lie groupoid $G \rightrightarrows M$ is *proper* if the map

$$(s,t) : G \longrightarrow M \times M$$

is proper.

In addition to the source cited above, [8] provides further background on the definitions used here.

2.2.2 Lie algebroids, regularity, and normal representations

Associated with a Lie groupoid $G \rightrightarrows M$ is its Lie algebroid

$$A_G := \ker(ds)|_M \longrightarrow M,$$

with anchor

$$\rho_G := dt|_{A_G} : A_G \to TM,$$

and bracket induced from right-invariant source-vertical vector fields [12, Sec. 6.1]. For each $x \in M$, the isotropy Lie algebra is

$$(\ker \rho_G)_x \cong \mathrm{Lie}(G_x).$$

Definition 2.8 ([11, 1.3(g), 1.5])**.** A Lie groupoid $G \rightrightarrows M$ is *regular* if its anchor

$$\rho_G : A_G \to TM$$

has locally constant rank; equivalently, the orbit dimension is locally constant.

If G is regular, then $\rho_G(A_G) \subseteq TM$ is a smooth involutive subbundle, and hence defines a foliation $\mathcal{O}$ on M whose leaves are the connected components of the orbits [11, 1.5].

If $O \subseteq M$ is an orbit, the restricted groupoid $G|_O \rightrightarrows O$ acts on the normal bundle

$$\nu(O) := TM|_O/TO$$

by the *normal representation*; dually, it acts on the conormal bundle [4, Sec. 2.2].

2.3 Riemannian groupoids

Following del Hoyo–Fernandes, if a Lie groupoid $G \rightrightarrows M$ acts on a manifold E, a Riemannian metric on E is called *transversely invariant* if the normal representation of the action groupoid

$$G \ltimes E \rightrightarrows E$$

acts by isometries [4, Definition 2.2.2].

Definition 2.9 ([4, Definitions 3.1.1, 3.2.1, 3.3.2])**.** Let $G \rightrightarrows M$ be a Lie groupoid.

1. A 0-*metric* on $G \rightrightarrows M$ is a Riemannian metric $\eta^{(0)}$ on $G^{(0)} = M$ that is transversely invariant for the canonical action $G \curvearrowright M$, given by $g \cdot s(g) = t(g)$.

2. A 1-*metric* on $G \rightrightarrows M$ is a Riemannian metric $\eta^{(1)}$ on $G^{(1)} = G$ that is transversely left-invariant and for which the inversion i is an isometry.

3. A 2-*metric* on $G \rightrightarrows M$ is a Riemannian metric $\eta^{(2)}$ on $G^{(2)}$ that is transversely invariant for the action

$$\theta_1 : G \curvearrowright G^{(2)}, \qquad k \cdot (h, g) := (kh, g),$$

and for which the group S_3 acts by isometries. The pair $(G \rightrightarrows M, \eta^{(2)})$ is called a *Riemannian groupoid*.

Here *transversely left-invariant* means transversely invariant for the left-translation action of G on G; equivalently, $\eta^{(1)}$ is s-transverse, and because inversion is an isometry, this is also equivalent to being t-transverse [4, Example 2.3.2 and Definition 3.2.1].

Proposition 2.10 ([4, Proposition 3.3.4])**.** *Let $G \rightrightarrows M$ be a Lie groupoid. A 2-metric $\eta^{(2)}$ on $G^{(2)}$ induces a 1-metric $\eta^{(1)}$ on $G^{(1)}$, and hence also a 0-metric $\eta^{(0)}$ on $G^{(0)}$.*

Theorem 2.11 (Del Hoyo–Fernandes, [4, Thm. 1])**.** *Every Hausdorff proper Lie groupoid admits a 2-metric.*

Chapter 3

Molino's structure theorem for regular Riemannian groupoids

This chapter develops Molino's structure theorem for regular Riemannian groupoids.

In the regular case, the orbit foliation provides the link with classical Molino's theory: the metric induced by the Riemannian groupoid structure makes the orbit foliation Riemannian, and passing to the transverse orthonormal frame bundle gives access to Molino's construction.

We then use Lie algebroids to organize the structural information coming from the leaf closures and to relate it to the isotropy and normal representation of the original groupoid.

3.1 Regular groupoids, metrics, and normal representations

3.1.1 Regular Riemannian groupoids and the orbit foliation

Following the discussion in Chapter 2, we start by discussing the tools we will need from [4] and [11].

Definition 3.1 ([11, 1.3(g), 1.5])**.** A Lie groupoid $G \rightrightarrows M$ is *regular* if the anchor map of its associated Lie algebroid $\mathfrak{g}$, $\rho : \mathfrak{g} \to TM$, has locally constant rank. Equivalently, for each $x \in M$, the target map $t : G \to M$ restricts to a map $t : s^{-1}(x) \to M$ of constant rank (if M is not connected, only locally of constant rank).

If this is the case, $T(\mathcal{O}) := \mathrm{im}(\rho) \subseteq TM$ is a smooth involutive subbundle, defining a regular foliation $\mathcal{O}$ on M whose leaves are the connected components of the orbits $O_x := t(s^{-1}(x))$.

We summarize these assumptions as our hypotheses.

Hypothesis 3.2. *Let $G \rightrightarrows M$ be a Hausdorff, regular Lie groupoid. Assume that M is compact and connected, and fix a 2-metric $\eta^{(2)}$ on $G^{(2)}$. Let $\mathcal{O}$ be the orbit foliation, and set $N := TM/T(\mathcal{O})$ with $\operatorname{rank} N = q = \operatorname{codim} \mathcal{O}$.*

Now we discuss Riemannian groupoids and induced transverse metrics. We follow del Hoyo–Fernandes [4]; in particular, the discussion below uses Sections 2 and 3 of that paper.

Let $G \rightrightarrows M$ be a Lie groupoid. We write $G^{(n)} \subseteq G^n$ for the manifold consisting of chains of n composable arrows,

$$G^{(0)} = M, \qquad G^{(1)} = G, \qquad G^{(2)} = G \times_M G = \{(h, g) \in G \times G \mid s(h) = t(g)\}.$$

In general,

$$G^{(n)} = \underbrace{G \times_M \cdots \times_M G}_{n \text{ factors}}.$$

On $G^{(2)}$, we denote by

$$\pi_1(h, g) = h, \qquad \pi_2(h, g) = g, \qquad m(h, g) = hg.$$

We now define the orbitwise normal representation and verify that it is well-defined and smooth.

Remark 3.3 (Normal representation). Let $G \rightrightarrows M$ be a Lie groupoid and let $O \subseteq M$ be an orbit. Write $G_O := G|_O = \{g \in G \mid s(g), t(g) \in O\} \rightrightarrows O$ for the restriction groupoid and denote the normal bundle of O in M by $\nu(O) := TM|_O / T(O)$. Let $\nu(G_O)$ denote the normal bundle of the immersed subgroupoid $G_O \subseteq G$, $\nu(G_O) := TG|_{G_O}/TG_O$.

Since $s|_{G_O}$ and $t|_{G_O}$ are submersions $G_O \to O$, ds and dt send $T(G_O)$ into $T(O)$ and hence descend to vector bundle maps $\overline{ds} : \nu(G_O) \to s^*\nu(O)$, $\overline{dt} : \nu(G_O) \to t^*\nu(O)$. Moreover, $\overline{ds}$ is a vector bundle isomorphism. Using the canonical identifications $s^*\nu(O) \cong G_O \times_O \nu(O)$ and $t^*\nu(O) \cong G_O \times_O \nu(O)$, the normal representation is the composite

$$G_O \times_O \nu(O) \cong s^*\nu(O) \xrightarrow{(\overline{ds})^{-1}} \nu(G_O) \xrightarrow{\overline{dt}} t^*\nu(O) \cong G_O \times_O \nu(O).$$

This means that for each arrow $g \in G_O$ with $s(g) = x$ and $t(g) = y$,

$$\lambda_g := \overline{dt}_g \circ (\overline{ds}_g)^{-1} : \nu_x(O) \to \nu_y(O).$$

We will return to this construction in Proposition 3.13. In particular, for a Lie groupoid action $\theta : G \curvearrowright E$, the associated action groupoid $G \times_M E \rightrightarrows E$ has a normal representation along each orbit in E.

Definition 3.4 ([4, Definition 2.2.2])**.** Let $\theta : G \curvearrowright E$ be a Lie groupoid action. A metric η^E on E is *transversely θ-invariant* if the normal representation of the action groupoid $G \times_M E \rightrightarrows E$ is by isometries, or equivalently, if the conormal representation is by isometries.

Our construction of the main theorem is based on the implication given by Proposition 2.10. The next step is to use the induced 0-metric to relate to the normal representation as in [4]. We summarize this in the following proposition.

Proposition 3.5. *Under Hypothesis 3.2, let $\eta^{(0)}$ be the metric on $M = G^{(0)}$ induced by the given 2-metric $\eta^{(2)}$, and let $g^M := \eta^{(0)}$. Then g^M is transversely invariant for the canonical action $G \curvearrowright M$, hence g^M is a 0-metric.*

Equivalently, for each orbit $O \subseteq M$, the normal representation $\lambda \colon G_O \curvearrowright \nu(O)$ acts by fibrewise isometries for the metric on $\nu(O)$ induced by g^M.

Proof. By Proposition 2.10, there is a 0-metric $\eta^{(0)}$ on M. By Definition 2.9, this means that $\eta^{(0)}$ is transversely invariant for the canonical action $G \curvearrowright M$. The second part follows directly from Definition 3.4. $\square$

Then we review the fact that the orbit foliation is Riemannian.

Proposition 3.6 ([4, Proposition 3.2.2(ii)]). *Under Hypothesis 3.2, the induced metric* $g^M := \eta^{(0)}$ *makes the orbit foliation* $\mathcal{O}$ *into a Riemannian foliation.*

3.1.2 Orbitwise and global normal representations

Recall that we briefly discussed the normal representation in Remark 3.3. Now we want to discuss the *orbitwise* normal representation.

Lemma 3.7. *Let* $G \rightrightarrows M$ *be a Lie groupoid and let* $x \in M$*. Then the restriction* $t_x := t|_{s^{-1}(x)} \colon s^{-1}(x) \to O_x := t\big(s^{-1}(x)\big)$ *is a surjective submersion and a principal* G_x*-bundle.*

Consequently, for any arrow $g \colon x \to y$*, we have* $dt_g(\ker ds_g) = T_y(O_x)$ *and* $ds_g(\ker dt_g) = T_x(O_x)$*.*

Proof. The first part is Theorem 5.4(iv) of [12]. For fixed $x \in M$,

$$t_x := t|_{s^{-1}(x)} \colon s^{-1}(x) \to O_x$$

is a principal G_x-bundle, hence a surjective submersion onto the orbit O_x.

The second part is also easy to show. Let $g \colon x \to y$. Since $s^{-1}(x) \subseteq G$ is a submanifold with $T_g\big(s^{-1}(x)\big) = \ker(ds_g)$, we have

$$d(t_x)_g = dt_g|_{\ker(ds_g)} \colon \ker(ds_g) \to T_y(O_x),$$

and it is surjective because t_x is a submersion. Therefore

$$dt_g(\ker ds_g) = T_y(O_x).$$

Using that the inversion map interchanges s and t, we get

$$ds_g(\ker dt_g) = T_x(O_x).$$

□

Definition 3.8 ([3, Prop. 3.4.2]; [4, Sec. 2.2])**.** Let $O \subseteq M$ be an orbit of the Lie groupoid $G \rightrightarrows M$, and let $g : x \to y$ be an arrow in G_O. For $[v] \in \nu_x(O) := T_x(M)/T_x(O)$, choose a representative $v \in T_x(M)$. Since $s : G \to M$ is a submersion, there exists $X \in T_g(G)$ such that $ds_g(X) = v$. This allows us to define

$$\lambda_g([v]) := [dt_g(X)] \in \nu_y(O) := T_y(M)/T_y(O),$$

where $[dt_g(X)]$ denotes the class of $dt_g(X)$ in the quotient. We call this the normal representation along the orbit.

We will show in Lemma 3.9 that this is independent of all choices.

Lemma 3.9. *For an arrow $g : x \to y$ in G_O,*

$$[dt_g(X)] \in \nu_y(O) = T_y(M)/T_y(O)$$

is independent of the choice of representative $v \in T_x(M)$ of $[v] \in \nu_x(O) = T_x(M)/T_x(O)$ and of the choice of lift $X \in T_g(G)$ with $ds_g(X) = v$. Hence

$$\lambda_g : \nu_x(O) \to \nu_y(O), \qquad \lambda_g([v]) := [dt_g(X)],$$

is a well-defined linear map.

Proof. From Lemma 3.7, we have

$$dt_g\big(\ker ds_g\big) = T_y(O), \tag{3.10}$$

$$ds_g\big(\ker dt_g\big) = T_x(O). \tag{3.11}$$

If $X, X' \in T_g(G)$ satisfy $ds_g(X) = ds_g(X') = v$, then $X' - X \in \ker(ds_g)$. By (3.10), $dt_g(X' - X) \in T_y(O)$, hence

$$[dt_g(X')] = [dt_g(X)] \in T_y(M)/T_y(O).$$

Suppose $v, v' \in T_x(M)$ represent the same class in $\nu_x(O) = T_x(M)/T_x(O)$, so $v' - v \in T_x(O)$. By (3.11), we can choose $Z \in \ker(dt_g)$ such that $ds_g(Z) = v' - v$. If X is a lift of v such that $ds_g(X) = v$, let $X' := X + Z$. Then

$$ds_g(X') = v' \qquad \text{and} \qquad dt_g(X') = dt_g(X).$$

This is to say λ_g is well-defined, and linearity is immediate from the construction. □

Lemma 3.12. *Let $O \subseteq M$ be an orbit and let $\lambda\colon G_O \curvearrowright \nu(O)$ be the normal representation. For the unit arrow 1_x, we have $\lambda_{1_x} = \mathrm{id}_{\nu_x(O)}$. For composable arrows $g\colon x \to y$ and $h\colon y \to z$ in G_O, we have $\lambda_{hg} = \lambda_h \circ \lambda_g$.*

Proof. This is covered in [4, Sec. 2.2]. □

Proposition 3.13 ([4, Sec. 2.2])**.** *For each orbit $O \subseteq M$, $\{\lambda_g\}_{g \in G_O}$ defines a smooth representation of the restriction Lie groupoid $G_O \rightrightarrows O$ on the vector bundle $\nu(O) \to O$.*

In particular, for each arrow $g\colon x \to y$ in G_O, the map $\lambda_g\colon \nu_x(O) \to \nu_y(O)$ is a linear isomorphism with inverse $\lambda_{g^{-1}}$.

Moreover, under the hypotheses of Proposition 3.5, each λ_g is an isometry for the metric on $\nu(O)$ induced by g^M [4, Def. 3.1.1].

Assume from now on that G is regular, so the orbit foliation $\mathcal{O}$ has constant rank and $N := T(M)/T(\mathcal{O}) \to M$ is a smooth vector bundle of rank q. We now consider the *global* normal representation on N.

Proposition 3.14. *The orbitwise normal representations of Proposition 3.13 assemble into a smooth representation $\lambda^N\colon G \times_M N \to N$, $(g, [v_x]) \mapsto \lambda^N_g([v_x])$, such that the bundle projection $\pi_N\colon N \to M$ satisfies $\pi_N(\lambda^N(g, [v_x])) = t(g)$. Equivalently, the following diagram commutes:*

$$\begin{array}{ccc} G \times_M N & \xrightarrow{\lambda^N} & N \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ G & \xrightarrow[t]{} & M. \end{array}$$

Under the hypotheses of Proposition 3.5, each λ^N_g is an isometry for the induced metric g^N on N.

Proof. For each arrow $g\colon x \to y$, let O_x denote the orbit through x (as well as through y). Because G is regular, we have $N_x = T_xM/T_x(\mathcal{O}) = \nu_x(O_x)$ and $N_y = T_yM/T_y(\mathcal{O}) = \nu_y(O_x)$. Let $\lambda^N_g\colon N_x \to N_y$ be the orbitwise normal map of Proposition 3.13 under these identifications.

We want to show smoothness. Consider $Q := TG/\ker(ds) \to G$. Since $s\colon G \to M$ is a submersion, ds induces a vector bundle isomorphism

$$\overline{ds}\colon Q \xrightarrow{\cong} s^*(TM).$$

We have used earlier that $dt_g(\ker ds_g) = T_y(\mathcal{O})$ for each $g\colon x \to y$, by Lemma 3.7. Hence dt descends to a smooth vector bundle map

$$\overline{dt}\colon Q \to t^*(N).$$

So far we can summarize the construction as follows:

$$\begin{array}{ccc} s^*TM & \xrightarrow{\overline{ds}^{-1}} & TG/\ker(ds) \\ \downarrow & & \downarrow \overline{dt} \\ s^*N & \overset{\lambda^N}{\dashrightarrow} & t^*N. \end{array}$$

It's interesting to note that $s^*(TM) \xrightarrow{\overline{ds}^{-1}} Q \xrightarrow{\overline{dt}} t^*(N)$ vanishes on $s^*(T(\mathcal{O}))$. If $v \in T_x(\mathcal{O})$, then by Lemma 3.7 there exists $\xi \in \ker(dt_g)$ with $ds_g(\xi) = v$, so the image of v in $t^*(N)$ is zero. Therefore the composite factors uniquely through $s^*(N) = s^*(TM)/s^*(T(\mathcal{O}))$, which yields a smooth vector bundle map

$$\lambda^N\colon s^*(N) \longrightarrow t^*(N).$$

Its fibre at g is λ_g^N. Equivalently, this is a smooth action map $\lambda^N\colon G \times_M N \to N$.

The last part follows from Proposition 3.5 quite directly. $\square$

3.2 Molino theory's frame bundle and lifted groupoid action

3.2.1 The transverse orthonormal frame bundle and lifted foliation

The following definitions and lemmas discuss Molino's frame bundle construction for the Riemannian foliation $(M, \mathcal{O})$, together with the Lie algebroid ingredients used in the proof of Theorem 3.41.

Our intention in keeping the functoriality and descent statements here is that they are used later in the corollaries and fit naturally with the frame bundle construction.

Definition 3.15 ([12, Ex. 4.19]). Let $q = \operatorname{rank} N$. The transverse orthonormal frame bundle of $(\mathcal{O}, g^N)$ is the principal $O(q)$-bundle $\pi : OF(M, \mathcal{O}) \to M$ whose fibre over $x \in M$ is

$$OF_x := \pi^{-1}(x) = \Big\{ e : \mathbb{R}^q \to N_x \mid e \text{ is an orthogonal linear isomorphism} \Big\},$$
$$\text{where } N_x \text{ is equipped with } g_x^N.$$

The right action of $O(q)$ is given by $e \cdot A := e \circ A$ for $A \in O(q)$.

To apply Molino's theory to the regular Riemannian foliation $(M, \mathcal{O})$, we recall the transverse orthonormal frame bundle $\pi\colon OF(M, \mathcal{O}) \to M$ and the associated Molino structures: the lifted foliation $\widetilde{\mathcal{O}}$, the transverse canonical form θ, and the transverse Levi-Civita connection ∇^{tr} on N.

Remark 3.16 ([12, Remark 2.7(2)]). Since $(M, \mathcal{O})$ is a Riemannian foliation with respect to g^M by Proposition 3.6, we may choose a Haefliger cocycle $(s_i\colon U_i \to T_i)$, where $T_i \subseteq \mathbb{R}^q$ is open, defining $\mathcal{O}$, such that each s_i is a submersion with connected fibres and $\ker(ds_i)_x = T_x(\mathcal{O})$ for $x \in U_i$.

On overlaps $U_i \cap U_j$, the transition maps $s_{ij}\colon s_j(U_i \cap U_j) \to s_i(U_i \cap U_j)$ satisfy $s_i = s_{ij} \circ s_j$.

Definition 3.17 ([12, Ex. 4.19])**.** Choose a Haefliger cocycle $(s_i\colon U_i \to T_i)$. Let $OF(T_i) \to T_i$ be the orthonormal frame bundle of the Riemannian manifold (T_i, g_i).

For $x \in U_i$, $(ds_i)_x\colon T_xM \to T_{s_i(x)}T_i$ has kernel $T_x(\mathcal{O})$, hence induces a linear isomorphism

$$(ds_i)_x^N : N_x = T_xM/T_x(\mathcal{O}) \longrightarrow T_{s_i(x)}T_i.$$

Let $\pi\colon OF(M, \mathcal{O}) \to M$ be the transverse orthonormal frame bundle and define

$$\widetilde{s}_i : \pi^{-1}(U_i) \longrightarrow OF(T_i), \qquad \widetilde{s}_i(e) := (ds_i)_{\pi(e)}^N \circ e.$$

For $f \in OF(T_i)$, the restriction of π identifies the fibre $\widetilde{s}_i^{-1}(f)$ diffeomorphically with the fibre $s_i^{-1}(\pi_i(f))$. In particular, each $\widetilde{s}_i$ is a submersion with connected fibres.

On overlaps $U_i \cap U_j$, define

$$\widetilde{s}_{ij} : OF(T_j)|_{s_j(U_i \cap U_j)} \longrightarrow OF(T_i)|_{s_i(U_i \cap U_j)}, \qquad \widetilde{s}_{ij}(f) := (ds_{ij})_y \circ f,$$

for $f \in OF(T_j)_y$. On $\pi^{-1}(U_i \cap U_j)$, we have

$$\widetilde{s}_i = \widetilde{s}_{ij} \circ \widetilde{s}_j.$$

Therefore the fibres of the $\widetilde{s}_i$ glue to a global foliation $\widetilde{\mathcal{O}}$ on $OF(M, \mathcal{O})$, which we call the lifted orbit foliation.

Thus $\pi : (OF(M,\mathcal{O}),\widetilde{\mathcal{O}}) \to (M,\mathcal{O})$ is a transverse principal $O(q)$-bundle.

Remark 3.18. We give some pictures. For each i, letting $\pi_{T_i} : OF(T_i) \to T_i$ be the orthonormal frame bundle projection, the local defining submersions of the lifted foliation fit into the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\widetilde{s}_i} & OF(T_i) \\ \pi\big\downarrow & & \big\downarrow\pi_{T_i} \\ U_i & \xrightarrow[s_i]{} & T_i. \end{array}$$

Thus the fibres of $\widetilde{s}_i$ project diffeomorphically to the fibres of s_i.

On overlaps $U_i \cap U_j$,

$$\begin{array}{ccc} \pi^{-1}(U_i\cap U_j) & \xrightarrow{\widetilde{s}_j} & OF(T_j)|_{s_j(U_i\cap U_j)} \\ & \searrow^{\widetilde{s}_i} & \big\downarrow\widetilde{s}_{ij} \\ & & OF(T_i)|_{s_i(U_i\cap U_j)}. \end{array}$$

Definition 3.19 ([12, Ex. 4.19]; [13, Sec. 2.4; Prop. 2.6; Sec. 3.3])**.** Let $\pi : OF(M,\mathcal{O}) \to M$ be the transverse orthonormal frame bundle, $\widetilde{\mathcal{O}}$ be the lifted orbit foliation, and $\mathrm{pr}^N : TM \longrightarrow N := TM/T(\mathcal{O})$ be the quotient projection. We define the following.

(i) The *transverse canonical form* is the 1-form $\theta \in \Omega^1(OF(M,\mathcal{O}),\mathbb{R}^q)$ defined by

$$\theta_e(\xi) := e^{-1}\left(\mathrm{pr}^N_{\pi(e)}\big((d\pi)_e(\xi)\big)\right), \qquad e \in OF(M,\mathcal{O}),\ \xi \in T_eOF(M,\mathcal{O}).$$

It is $O(q)$-equivariant and satisfies $\ker(\theta_e) = \ker\big((d\pi)_e\big) \oplus T_e(\widetilde{\mathcal{O}})$.

(ii) For each i, let $(ds_i)^N : N|_{U_i} \longrightarrow s_i^*(TT_i)$ be the induced isometric bundle isomorphism, and let ∇^i be the ordinary Levi–Civita connection of the Riemannian manifold (T_i, g_i). Define a connection on $N|_{U_i}$ by

$$\nabla^{\mathrm{tr},i} := \big((ds_i)^N\big)^{-1} \circ s_i^*\nabla^i \circ (ds_i)^N.$$

The $\nabla^{\mathrm{tr},i}$ can be glued to a global metric connection ∇^{tr} on N, which we call the *transverse Levi–Civita connection.*

(iii) Let $\varepsilon_1, \dots, \varepsilon_q$ be the standard basis of $\mathbb{R}^q$. The tautological orthonormal frame $E_1, \dots, E_q$ of π^*N is defined by $E_a(e) := e(\varepsilon_a)$, $e \in OF(M, \mathcal{O})$. The *transverse Levi–Civita connection form* is the unique 1-form $\omega \in \Omega^1(OF(M, \mathcal{O}), \mathfrak{o}(q))$ characterized by

$$(\pi^*\nabla^{\mathrm{tr}})_\xi E_a = \sum_{b=1}^{q} \omega_{ba}(\xi)\, E_b, \qquad \xi \in T_e OF(M, \mathcal{O}).$$

Since ∇^{tr} is metric and $E_1, \dots, E_q$ is orthonormal, the matrix (ω_{ba}) is skew-symmetric.

Proposition 3.20. *The local connections $\nabla^{\mathrm{tr},i}$ are compatible on overlaps and determine a global metric connection ∇^{tr} on N, independent of the chosen Haefliger cocycle. If $\pi_{T_i} \colon OF(T_i) \to T_i$ denotes the orthonormal frame bundle of (T_i, g_i) and ω_i denotes its ordinary Levi–Civita connection form, then on $\pi^{-1}(U_i)$, $\omega = \widetilde{s}_i^{\,*}\omega_i$.*

In particular, ω is a projectable connection on the transverse principal $O(q)$-bundle $(OF(M, \mathcal{O}), \widetilde{\mathcal{O}})$.

Proof. Following Definition 3.19, on $U_i \cap U_j$, the transition map $s_{ij} : s_j(U_i \cap U_j) \to s_i(U_i \cap U_j)$ is a local isometry. Since a local isometry is an isometry onto an open subset, the naturality theorem for the Levi–Civita connection applies after restriction [7, Proposition 5.13]. Hence $s_{ij}^*\nabla^i = \nabla^j$ on $s_j(U_i \cap U_j)$. Because $(ds_i)^N = (ds_{ij}) \circ (ds_j)^N$ on $U_i \cap U_j$, the pullback connections agree:

$$\nabla^{\mathrm{tr},i} = \nabla^{\mathrm{tr},j} \qquad \text{on } U_i \cap U_j.$$

Therefore the $\nabla^{\mathrm{tr},i}$ glues to a global metric connection ∇^{tr} on N.

To prove independence of the chosen cocycle, let $(s'_a : U'_a \to T'_a)$ be another Haefliger cocycle and let ∇'_a be the Levi–Civita connection on TT'_a. For $x \in U_i \cap U'_a$, after shrinking if necessary, there is an open neighbourhood $W \subseteq U_i \cap U'_a$ of x and a local diffeomorphism $h_{ia} : s'_a(W) \longrightarrow s_i(W)$ such that

$$s_i|_W = h_{ia} \circ s'_a|_W.$$

Since both $(ds_i)^N$ and $(ds'_a)^N$ are fibrewise isometries, h_{ia} is a local isometry. Applying again [7, Proposition 5.13], we obtain

$$h_{ia}^* \nabla^i = \nabla'_a.$$

Hence the corresponding local pullback connections agree on W, so the global connection ∇^{tr} is independent of the cocycle.

It is clear that the lifted map $\widetilde{s}_i : \pi^{-1}(U_i) \to OF(T_i)$ identifies $\pi^* N|_{\pi^{-1}(U_i)}$ with $\widetilde{s}_i^*(\pi_{T_i}^* TT_i)$, and the tautological frame $E_1, \ldots, E_q$ on $\pi^* N$ corresponds to the tautological frame $E_1^i, \ldots, E_q^i$ on $\pi_{T_i}^* TT_i$. Hence, the pullback connection $\pi^* \nabla^{\mathrm{tr}}$ corresponds to $\widetilde{s}_i^*(\pi_{T_i}^* \nabla^i)$, and by the defining property of connection forms, $\omega|_{\pi^{-1}(U_i)} = (\widetilde{s}_i)^* \omega_i$.

Finally, on $\pi^{-1}(U_i)$, the foliation $\widetilde{\mathcal{O}}$ is the foliation by fibres of $\widetilde{s}_i$. Hence, for every $X \in \mathfrak{X}(\widetilde{\mathcal{O}})$, $i_X \omega = i_X (\widetilde{s}_i)^* \omega_i = 0$ and $i_X(d\omega) = i_X(\widetilde{s}_i)^*(d\omega_i) = 0$, so Cartan's formula gives

$$L_X \omega = d(i_X \omega) + i_X(d\omega) = 0.$$

Thus ω is a projectable connection on the transverse principal $O(q)$-bundle $(OF(M, \mathcal{O}), \widetilde{\mathcal{O}})$.

□

Lemma 3.21 ([12, Theorem 4.20])**.** *Let $\pi : OF(M, \mathcal{O}) \to M$ be the transverse orthonormal frame bundle of a Riemannian foliation $(M, \mathcal{O})$ of codimension q, with lifted foliation $\widetilde{\mathcal{O}}$, transverse canonical form θ, and transverse Levi–Civita connection form ω.*

Then, (θ, ω) induces a vector bundle isomorphism

$$\overline{\Psi} : N(\widetilde{\mathcal{O}}) \longrightarrow OF(M, \mathcal{O}) \times \big(\mathbb{R}^q \oplus \mathfrak{o}(q)\big), \qquad [\xi] \longmapsto \big(\theta(\xi), \omega(\xi)\big),$$

where $N(\widetilde{\mathcal{O}}) = TOF(M, \mathcal{O})/T(\widetilde{\mathcal{O}})$. In particular, $(OF(M, \mathcal{O}), \widetilde{\mathcal{O}})$ is transversely parallelizable.

Proof. For each $e \in OF(M, \mathcal{O})$, define

$$\Psi_e : T_e OF(M, \mathcal{O})/T_e(\widetilde{\mathcal{O}}) \longrightarrow \mathbb{R}^q \oplus \mathfrak{o}(q), \qquad [\xi] \longmapsto \big(\theta_e(\xi), \omega_e(\xi)\big).$$

First, by Definition 3.19(i) we have $\theta|_{T(\widetilde{\mathcal{O}})} = 0$, and by Proposition 3.20 we also have $\omega|_{T(\widetilde{\mathcal{O}})} = 0$. Hence Ψ_e is well-defined.

Recall that $\ker(\theta_e) = \ker\big((d\pi)_e\big) \oplus T_e(\widetilde{\mathcal{O}})$. So let $[\xi] \in T_e OF(M, \mathcal{O})/T_e(\widetilde{\mathcal{O}})$ satisfy $\Psi_e([\xi]) = 0$. Choose $\xi \in T_e OF(M, \mathcal{O})$ with $\theta_e(\xi) = 0$ and $\omega_e(\xi) = 0$. Since $\theta_e(\xi) = 0$, it makes sense that $\xi = v + \xi_{\widetilde{\mathcal{O}}}$, $v \in \ker\big((d\pi)_e\big)$, $\xi_{\widetilde{\mathcal{O}}} \in T_e(\widetilde{\mathcal{O}})$.

Because ω vanishes on $T(\widetilde{\mathcal{O}})$, we have $0 = \omega_e(\xi) = \omega_e(v) + \omega_e(\xi_{\widetilde{\mathcal{O}}}) = \omega_e(v)$. Since ω is a connection form on the principal $O(q)$-bundle $\pi : OF(M, \mathcal{O}) \to M$, its restriction $\omega_e : \ker\big((d\pi)_e\big) \longrightarrow \mathfrak{o}(q)$ to the vertical tangent space is a linear isomorphism. Hence $v = 0$, so $\xi \in T_e(\widetilde{\mathcal{O}})$, and therefore $[\xi] = 0$. Thus Ψ_e is injective.

Finally, $\dim N_e(\widetilde{\mathcal{O}}) = q + \dim \mathfrak{o}(q) = \dim\big(\mathbb{R}^q \oplus \mathfrak{o}(q)\big)$, so Ψ_e is an isomorphism. Since the construction is smooth in e, the maps Ψ_e assemble to a smooth vector bundle isomorphism $\overline{\Psi}$. □

Definition 3.22. A local diffeomorphism $\varphi : U \to V$ with $U, V \subseteq M$ open is called a foliated transverse isometry if:

(i) φ is foliated, i.e. it sends leaves of $\mathcal{O}|_U$ to leaves of $\mathcal{O}|_V$. Equivalently, $d\varphi_x\big(T_x(\mathcal{O})\big) = T_{\varphi(x)}(\mathcal{O})$ for all $x \in U$. In this case φ induces a well-defined linear map

$$(d\varphi)^N_x : \ N_x = T_x(M)/T_x(\mathcal{O}) \longrightarrow N_{\varphi(x)} = T_{\varphi(x)}(M)/T_{\varphi(x)}(\mathcal{O}),$$
$$(d\varphi)^N_x([v]) := [d\varphi_x(v)].$$

(ii) For every $x \in U$, the induced map $(d\varphi)^N_x$ is an isometry for g^N. We write

$$g^N_{\varphi(x)}\big((d\varphi)^N_x(u), (d\varphi)^N_x(v)\big) = g^N_x(u, v), \qquad u, v \in N_x.$$

We summarize the naturality of the transverse orthonormal frame bundle, its lifted foliation, and the forms θ and ω under foliated transverse isometries.

Lemma 3.23. *Let $\pi : OF(M, \mathcal{O}) \to M$ be the transverse orthogonal frame bundle, with lifted foliation $\widetilde{\mathcal{O}}$, transverse canonical form θ, transverse Levi–Civita connection ∇^{tr} on N, and associated principal connection form ω on $OF(M, \mathcal{O})$. If $\varphi : U \to V$ is a foliated transverse isometry, then*

$$OF(\varphi) : OF(M, \mathcal{O})|_U = \pi^{-1}(U) \to OF(M, \mathcal{O})|_V = \pi^{-1}(V),$$

defined by $OF(\varphi)(e) := (d\varphi)^N_{\pi(e)} \circ e$ is a principal $O(q)$-bundle isomorphism such that $\pi \circ OF(\varphi) = \varphi \circ \pi$, and

$$OF(\varphi)^*\big(\theta|_{\pi^{-1}(V)}\big) = \theta|_{\pi^{-1}(U)},$$
$$OF(\varphi)^*\big(\omega|_{\pi^{-1}(V)}\big) = \omega|_{\pi^{-1}(U)},$$
$$dOF(\varphi)\big(T(\widetilde{\mathcal{O}})|_{\pi^{-1}(U)}\big) = T(\widetilde{\mathcal{O}})|_{\pi^{-1}(V)}.$$

Proof. For each $x \in U$, the induced linear map $(d\varphi)_x^N\colon N_x \to N_{\varphi(x)}$ is an isometry by Definition 3.22(ii). Thus, if $e\colon \mathbb{R}^q \to N_x$ is an orthonormal frame, then

$$(d\varphi)_x^N \circ e\colon \mathbb{R}^q \to N_{\varphi(x)}$$

is again an orthonormal frame. Therefore $OF(\varphi)$ is well-defined.

It is $O(q)$-equivariant by construction because

$$\begin{aligned} OF(\varphi)(e \cdot A) &= (d\varphi)_{\pi(e)}^N \circ (e \circ A) \\ &= \big((d\varphi)_{\pi(e)}^N \circ e\big) \circ A \\ &= OF(\varphi)(e) \cdot A. \end{aligned}$$

Since φ is a local diffeomorphism, the same construction applies to φ^{-1}; hence $OF(\varphi)$ is a principal $O(q)$-bundle isomorphism.

Let $e \in OF(M, \mathcal{O})|_U$ and $\xi \in T_eOF(M, \mathcal{O})$, and write $x := \pi(e)$. Using $\pi \circ OF(\varphi) = \varphi \circ \pi$, the definition of θ, and the identity $\mathrm{pr}_{\varphi(x)}^N \circ d\varphi_x = (d\varphi)_x^N \circ \mathrm{pr}_x^N$ (which clearly holds because φ is foliated), we compute

$$\begin{aligned} \big(OF(\varphi)^*\theta\big)_e(\xi) &= \theta_{OF(\varphi)(e)}\big(dOF(\varphi)_e(\xi)\big) \\ &= OF(\varphi)(e)^{-1}\left(\mathrm{pr}_{\varphi(x)}^N\Big(d\pi_{OF(\varphi)(e)}\big(dOF(\varphi)_e(\xi)\big)\Big)\right) \\ &= OF(\varphi)(e)^{-1}\left(\mathrm{pr}_{\varphi(x)}^N\Big(d(\pi \circ OF(\varphi))_e(\xi)\Big)\right) \\ &= OF(\varphi)(e)^{-1}\left((d\varphi)_x^N\Big(\mathrm{pr}_x^N\big(d\pi_e(\xi)\big)\Big)\right) \\ &= e^{-1}\left(\mathrm{pr}_x^N\left(d\pi_e(\xi)\right)\right) \\ &= \theta_e(\xi), \end{aligned}$$

since $OF(\varphi)(e) = (d\varphi)_x^N \circ e$. Thus,

$$OF(\varphi)^*\big(\theta|_{\pi^{-1}(V)}\big) = \theta|_{\pi^{-1}(U)}.$$

Then we choose a Haefliger cocycle $(s_i\colon U_i \to T_i)$. For each i, the lifted foliation $\widetilde{\mathcal{O}}$ on $\pi^{-1}(U_i)$ is the foliation by fibres of the submersion $\widetilde{s}_i\colon \pi^{-1}(U_i) \to OF(T_i)$. Since φ is foliated, after shrinking if necessary, we may assume that for each i there exists j and a local diffeomorphism $h_{ji}\colon s_i(U \cap U_i) \to s_j(V \cap U_j)$ such that $s_j \circ \varphi = h_{ji} \circ s_i$ on $U \cap U_i$.

Because φ is a transverse isometry, each h_{ji} is a local isometry between the Riemannian manifolds (T_i, g_i) and (T_j, g_j). Then

$$\widetilde{s}_j \circ OF(\varphi) = OF(h_{ji}) \circ \widetilde{s}_i \quad \text{on } \pi^{-1}(U \cap U_i).$$

Hence $OF(\varphi)$ sends fibres of $\widetilde{s}_i$ to fibres of $\widetilde{s}_j$, so it preserves the lifted foliation $\widetilde{\mathcal{O}}$.

That is, the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U \cap U_i) & \xrightarrow{OF(\varphi)} & \pi^{-1}(V \cap U_j) \\ \downarrow{\scriptstyle \widetilde{s}_i} & & \downarrow{\scriptstyle \widetilde{s}_j} \\ OF(T_i) & \xrightarrow[OF(h_{ji})]{} & OF(T_j) \end{array}$$

Let ω_i and ω_j be the ordinary Levi–Civita connection forms on $OF(T_i)$ and $OF(T_j)$. By Proposition 3.20,

$$\omega|_{\pi^{-1}(U_i)} = (\widetilde{s}_i)^*\omega_i, \qquad \omega|_{\pi^{-1}(U_j)} = (\widetilde{s}_j)^*\omega_j.$$

Since h_{ji} is a local isometry, the naturality theorem applies to the Levi–Civita connection, as we discussed earlier [7, Proposition 5.13]. Equivalently, on orthonormal frame bundles the induced map preserves the Levi–Civita connection forms: $OF(h_{ji})^*\omega_j = \omega_i$.

Therefore, on $\pi^{-1}(U \cap U_i)$, it is easy to compute that

$$\begin{aligned} OF(\varphi)^*\omega &= OF(\varphi)^*(\widetilde{s}_j)^*\omega_j \\ &= (\widetilde{s}_i)^* OF(h_{ji})^*\omega_j \\ &= (\widetilde{s}_i)^*\omega_i \\ &= \omega. \end{aligned}$$

Hence

$$OF(\varphi)^*\big(\omega|_{\pi^{-1}(V)}\big) = \omega|_{\pi^{-1}(U)}.$$

Since $OF(\varphi)$ preserves $\widetilde{\mathcal{O}}$, we also have

$$dOF(\varphi)\big(T(\widetilde{\mathcal{O}})|_{\pi^{-1}(U)}\big) = T(\widetilde{\mathcal{O}})|_{\pi^{-1}(V)}.$$

□

3.2.2 Lifted groupoid action and invariance

We now use the global normal representation (Proposition 3.14) to lift the action of G on M to an action on the transverse orthonormal frame bundle $\pi : OF(M, \mathcal{O}) \to M$.

Definition 3.24. Let $\lambda^N : G \curvearrowright N$ be the global normal representation on $N = T(M)/T(\mathcal{O})$. For an arrow $g \in G$ with $s(g) = x$ and $t(g) = y$, and for a frame $e \in OF_x := \pi^{-1}(x)$, define $g \cdot e := \lambda^N_g \circ e \in OF_y := \pi^{-1}(y)$. This is well-defined since λ^N_g is an isometry for g^N.

Equivalently, this defines a map $G \times_M OF(M, \mathcal{O}) \to OF(M, \mathcal{O})$, $(g, e) \mapsto g \cdot e$, where $G \times_M OF(M, \mathcal{O}) := \{(g, e) \in G \times OF(M, \mathcal{O}) \mid s(g) = \pi(e)\}$.

Now we discuss the lifted groupoid action on OF.

Proposition 3.25. *From Definition 3.24, there is a smooth left action of G on the principal $O(q)$-bundle $\pi : OF(M, \mathcal{O}) \to M$. In particular, for all composable arrows g, h and all frames e, we have $h \cdot (g \cdot e) = (hg) \cdot e$ and $1_x \cdot e = e$, and the action commutes with the right $O(q)$-action: $g \cdot (eA) = (g \cdot e)A$ for $A \in O(q)$.*

Proof. Well-definedness is clear. Since each $\lambda_g^N : N_x \to N_y$ is a linear isomorphism, $\lambda_g^N \circ e$ is a linear isomorphism $\mathbb{R}^q \to N_y$. Moreover, λ_g^N is an isometry with respect to the metric g^N, hence $\lambda_g^N \circ e$ is again an orthogonal frame, so $g \cdot e \in OF_y$.

We now show that $\mu : G \times_M OF(M, \mathcal{O}) \longrightarrow OF(M, \mathcal{O})$, where $(g, e) \mapsto \lambda_g^N \circ e$, is smooth. Let $U, V \subseteq M$ be open sets and choose smooth local sections $b_U : U \to OF(M, \mathcal{O})$ and $b_V : V \to OF(M, \mathcal{O})$. They give local trivializations of the principal $O(q)$-bundle $\pi : OF(M, \mathcal{O}) \to M$ such that $\psi_U : U \times O(q) \xrightarrow{\cong} \pi^{-1}(U)$, $\psi_U(x, A) = b_U(x) \cdot A = b_U(x) \circ A$, and similarly $\psi_V : V \times O(q) \xrightarrow{\cong} \pi^{-1}(V)$.

Consider the open set $G_U^V := \{g \in G \mid s(g) \in U,\ t(g) \in V\} \subseteq G$. Define a map $L : G_U^V \to O(q)$ by $L(g) := b_V\big(t(g)\big)^{-1} \circ \lambda_g^N \circ b_U\big(s(g)\big) \in O(q)$. This is well defined because $b_U(s(g))$ and $b_V(t(g))$ are orthogonal frames and λ_g^N is an isometry, hence the above composite is an orthogonal linear map $\mathbb{R}^q \to \mathbb{R}^q$. Moreover, by Proposition 3.14, L is smooth since $\lambda^N : G \times_M N \to N$ is smooth and b_U, b_V are smooth.

Now take $g \in G_U^V$ and write $x := s(g)$. For $A \in O(q)$ we have

$$\mu\big(g, \psi_U(x, A)\big) = \lambda_g^N \circ (b_U(x) \circ A) = \big(\lambda_g^N \circ b_U(x)\big) \circ A = \big(b_V(t(g)) \circ L(g)\big) \circ A = \psi_V\big(t(g), L(g)A\big).$$

Therefore, in the local coordinates given by ψ_U and ψ_V, the action map is $(g,(x,A)) \longmapsto \big(t(g), L(g)A\big)$, which is smooth. Since such trivializations cover M, it follows that μ is smooth globally.

The remaining claims follow from Proposition 3.14, together with the definition of the right action $e \cdot A = e \circ A$ on $OF(M,\mathcal{O})$. $\square$

Lemma 3.26. *Let $G \rightrightarrows M$ be a regular Lie groupoid and let $\lambda^N : G \curvearrowright N$ be the normal representation. Suppose G acts on $OF(M,\mathcal{O}) \to M$ by principal $O(q)$-bundle isomorphisms over the action on M. This is to say there is a smooth map $G \times_M OF(M,\mathcal{O}) \to OF(M,\mathcal{O})$, $(g,e) \mapsto g \cdot e$.*

Assume that the induced action on the associated bundle $OF(M,\mathcal{O}) \times_{O(q)} \mathbb{R}^q \cong N$ equals λ^N. Then, for every arrow $g : x \to y$ and every frame $e \in OF_x := \pi^{-1}(x)$, we have $g \cdot e = \lambda^N_g \circ e$.

In particular, the action from Definition 3.24 is the unique lift of λ^N to an action on $OF(M,\mathcal{O})$.

Proof. Let $\Phi : OF(M,\mathcal{O}) \times_{O(q)} \mathbb{R}^q \to N$ be the canonical identification. That is, $\Phi([e,u]) = e(u) \in N_{\pi(e)}$, where the equivalence relation is $(eA,u) \sim (e,Au)$. For $u \in \mathbb{R}^q$, by definition of the induced action on the associated bundle we have $g \cdot [e,u] = [g \cdot e, u]$. It's easy to check that $(g \cdot e)(u) = \lambda^N_g(e(u))$. Since this holds for all $u \in \mathbb{R}^q$, it follows that $g \cdot e = \lambda^N_g \circ e$. $\square$

Following Lemma 3.26, we obtain the left action groupoid $\mathcal{H} := G \ltimes OF(M,\mathcal{O}) \rightrightarrows OF(M,\mathcal{O})$ with manifold of arrows $\mathcal{H}_1 = G \times_M OF(M,\mathcal{O}) = \{(g,e) \in G \times OF(M,\mathcal{O}) \mid s(g) = \pi(e)\}$. We have structure maps $s_{\mathcal{H}}(g,e) = e$ and $t_{\mathcal{H}}(g,e) = g \cdot e$. Composition is given by $(h, g \cdot e) \circ (g,e) = (hg, e)$, units by $u_{\mathcal{H}}(e) = (1_{\pi(e)}, e)$, and inversion by $(g,e)^{-1} = (g^{-1}, g \cdot e)$.

Proposition 3.27. *Assume Hypothesis 3.2 and assume in addition that $G \rightrightarrows M$ is proper. Then the action groupoid $\mathcal{H} := G \ltimes OF(M,\mathcal{O}) \rightrightarrows OF(M,\mathcal{O})$ is a proper Lie groupoid.*

Proof. We want to show that the anchor map $(s_{\mathcal{H}}, t_{\mathcal{H}}) : \mathcal{H}_1 \to OF(M, \mathcal{O}) \times OF(M, \mathcal{O})$ is proper.

Recall that $\mathcal{H}_1 = G \times_M OF(M, \mathcal{O}) = \{(g, e) \in G \times OF(M, \mathcal{O}) \mid s(g) = \pi(e)\}$ and that $s_{\mathcal{H}}(g, e) = e$, $t_{\mathcal{H}}(g, e) = g \cdot e$. Let $p : \mathcal{H}_1 \to G$, $p(g, e) = g$ be the projection.

Since $\mathcal{H}_1 \cong s^*OF(M, \mathcal{O})$ as a pullback of the principal $O(q)$-bundle $\pi : OF(M, \mathcal{O}) \to M$, p is a principal $O(q)$-bundle. Because $O(q)$ is compact, p is proper.

Next, we have $(\pi \times \pi) \circ (s_{\mathcal{H}}, t_{\mathcal{H}}) = (s, t) \circ p$. For $(g, e) \in \mathcal{H}_1$ we have $\pi(e) = s(g)$ by definition of the fibre product, and also $\pi\big(t_{\mathcal{H}}(g, e)\big) = \pi(g \cdot e) = t(g)$.

Since G is proper, $(s, t) : G \to M \times M$ is proper. As p is proper, the composition $(\pi \times \pi) \circ (s_{\mathcal{H}}, t_{\mathcal{H}}) = (s, t) \circ p : \mathcal{H}_1 \to M \times M$ is proper.

Now let $K \subseteq OF(M, \mathcal{O}) \times OF(M, \mathcal{O})$ be compact. Then $(\pi \times \pi)(K) \subseteq M \times M$ is compact, and hence $\big((\pi \times \pi) \circ (s_{\mathcal{H}}, t_{\mathcal{H}})\big)^{-1}\big((\pi \times \pi)(K)\big)$ is compact.

Since $OF(M, \mathcal{O}) \times OF(M, \mathcal{O})$ is Hausdorff, K is closed, so $(s_{\mathcal{H}}, t_{\mathcal{H}})^{-1}(K)$ is a closed subset of this compact set. Therefore $(s_{\mathcal{H}}, t_{\mathcal{H}})^{-1}(K)$ is compact, and $(s_{\mathcal{H}}, t_{\mathcal{H}})$ is proper. □

Corollary 3.28. *Under the hypotheses of Proposition 3.27, the action groupoid* $\mathcal{H} := G \ltimes OF(M, \mathcal{O}) \rightrightarrows OF(M, \mathcal{O})$ *is Hausdorff and proper. In particular,* $\mathcal{H}$ *admits a* 2*-metric.*

Proof. Since G is Hausdorff and $OF(M, \mathcal{O})$ is Hausdorff, the arrow manifold $\mathcal{H}_1 = G \times_M OF(M, \mathcal{O})$ is a submanifold of the Hausdorff manifold $G \times OF(M, \mathcal{O})$, hence Hausdorff. Therefore $\mathcal{H}$ is Hausdorff and proper.

By Theorem 2.11, every Hausdorff proper Lie groupoid admits a 2-metric, so $\mathcal{H}$ admits a 2-metric. □

We next discuss bisections, the normal representation, and invariance of Molino's theory.

Definition 3.29 ([12, Sec. 5.1])**.** Let $G \rightrightarrows M$ be a Lie groupoid. A *local bisection* is a smooth map $\sigma : U \to G$ defined on an open set $U \subseteq M$ such that $s \circ \sigma = \mathrm{id}_U$ and $t \circ \sigma : U \to t(\sigma(U))$ is a diffeomorphism onto an open subset.

For a local bisection $\sigma : U \to G$, we write

$$f_\sigma : \pi^{-1}(U) \to \pi^{-1}(t(\sigma(U))), \qquad f_\sigma(e) := \sigma(\pi(e)) \cdot e,$$

for the induced principal $O(q)$-bundle isomorphism.

Next we study the local diffeomorphisms induced by local bisections. We first show that they preserve the orbit foliation, and then identify their induced normal maps with the normal representation.

Lemma 3.30. *Let $\sigma : U \to G$ be a local bisection of G and set $\varphi = t \circ \sigma : U \to V$. Then φ is orbit-preserving. For any $x_1, x_2 \in U$ in the same G-orbit, $\varphi(x_1), \varphi(x_2) \in V$ lie in the same G-orbit. In particular, since φ is a diffeomorphism, it maps each leaf of the orbit foliation $\mathcal{O}$ in U diffeomorphically onto a leaf of $\mathcal{O}$ in V, and hence for all $x \in U$ we have $d\varphi_x\big(T_x(\mathcal{O})\big) = T_{\varphi(x)}(\mathcal{O})$.*

Proof. Let $x \in U$, and let O_x be the orbit through x. Since σ is a local bisection, the arrow $\sigma(x) \in G$ has source $s(\sigma(x)) = x$ and target $t(\sigma(x)) = \varphi(x)$. Hence x and $\varphi(x)$ lie in the same orbit, so $O_{\varphi(x)} = O_x$ and φ preserves orbits.

Now let L_x denote the leaf of the orbit foliation $\mathcal{O}$ through x, i.e., the connected component of O_x containing x. Since φ is a homeomorphism, $\varphi(L_x)$ is connected and contains $\varphi(x)$, and we already know $\varphi(L_x) \subseteq O_{\varphi(x)} = O_x$. Then $\varphi(L_x) \subseteq L_{\varphi(x)}$, where $L_{\varphi(x)}$ is the connected component of O_x containing $\varphi(x)$. Applying the same argument to φ^{-1}, we obtain $\varphi^{-1}(L_{\varphi(x)}) \subseteq L_x$. Applying φ to this inclusion gives $L_{\varphi(x)} \subseteq \varphi(L_x)$.

Thus, $\varphi(L_x) = L_{\varphi(x)}$, so φ sends leaves of $\mathcal{O}$ to leaves of $\mathcal{O}$. Equivalently, φ is foliated for the orbit foliation $\mathcal{O}$. □

Here we introduce a bridge lemma; our intention is to identify the induced map on the normal bundle with the global normal representation.

Lemma 3.31. *Let $G \rightrightarrows M$ be regular with orbit foliation $\mathcal{O}$ and normal bundle $N = T(M)/T(\mathcal{O})$. Let $\sigma : U \to G$ be a local bisection and $\varphi := t \circ \sigma : U \to V$. Then for every $x \in U$ we have $(d\varphi)^N_x = \lambda^N_{\sigma(x)} : N_x \to N_{\varphi(x)}$.*

Proof. Let $x \in U$ and $v \in T_x(M)$, and let $X = d\sigma_x(v) \in T_{\sigma(x)}(G)$. Since $s \circ \sigma = \mathrm{id}_U$, we have $ds_{\sigma(x)}(X) = v$.

Let O be the orbit through x. Under regularity, we identify $N_x = T_x(M)/T_x(\mathcal{O}) = \nu_x(O)$ and similarly $N_{\varphi(x)} = \nu_{\varphi(x)}(O)$. Recall from Proposition 3.14 that λ^N is the orbitwise normal representation, so Definition 3.8 gives $\lambda^N_{\sigma(x)}([v]) = [dt_{\sigma(x)}(X)]$.

But $dt_{\sigma(x)}(X) = d(t \circ \sigma)_x(v) = d\varphi_x(v)$, hence $\lambda^N_{\sigma(x)}([v]) = [d\varphi_x(v)]$. By Lemma 3.30, φ is foliated, so $[d\varphi_x(v)] = (d\varphi)^N_x([v])$. □

Remark 3.32 (Clarification). For a non-étale Lie groupoid, an arrow $g : x \to y$ does not determine a unique germ of a local diffeomorphism of M near x: different local bisections σ through g may yield different maps $\varphi = t \circ \sigma$. Lemma 3.31 shows that for any such local bisection we have $(d\varphi)^N_x = \lambda^N_g : N_x \to N_y$. So the induced normal map depends only on the arrow g, not on the choice of local bisection.

Proposition 3.33. *Under Hypothesis 3.2, the lifted foliation $\widetilde{\mathcal{O}}$, the transverse canonical form θ, and the transverse Levi–Civita connection form ω are invariant under the local diffeomorphisms of $OF(M, \mathcal{O})$ induced by local bisections of G. Explicitly, for each local bisection $\sigma : U \to G$ with $V = t(\sigma(U))$, the induced map*

$$f : \pi^{-1}(U) \to \pi^{-1}(V)$$

satisfies

$$f^*\big(\theta|_{\pi^{-1}(V)}\big) = \theta|_{\pi^{-1}(U)}, \qquad f^*\big(\omega|_{\pi^{-1}(V)}\big) = \omega|_{\pi^{-1}(U)}, \qquad (df)\big(T(\widetilde{\mathcal{O}})|_{\pi^{-1}(U)}\big) = T(\widetilde{\mathcal{O}})|_{\pi^{-1}(V)}.$$

Proof. Let $\sigma : U \to G$ be a local bisection and let $\varphi := t \circ \sigma : U \to V$. By Lemma 3.30, φ preserves the orbit foliation. By Lemma 3.31, $(d\varphi)^N_x = \lambda^N_{\sigma(x)}$ on the normal bundles. By Proposition 3.14, the maps λ^N_g are fibrewise isometries for g^N. Hence φ is a foliated transverse isometry in the sense of Definition 3.22.

We compare the two induced maps on frames. For $e \in OF_x$ with $x = \pi(e)$,

$$f(e) = \sigma(x) \cdot e = \lambda^N_{\sigma(x)} \circ e = (d\varphi)^N_x \circ e = OF(\varphi)(e).$$

Hence $f = OF(\varphi)$, and the conclusion follows from Lemma 3.23. □

We give the following brief discussion of a consequence of the transverse principal bundle structure.

Proposition 3.34. *Let $(Q, \mathcal{F})$ be a foliated manifold, and let $\pi : \widehat{M} \to Q$ be a principal $O(q)$-bundle. Assume that $\widehat{M}$ carries a foliation $\widehat{\mathcal{F}}$ such that $\pi : (\widehat{M}, \widehat{\mathcal{F}}) \to (Q, \mathcal{F})$ is a transverse principal $O(q)$-bundle. Let $L_{\widehat{M}}$ be a leaf of $\widehat{\mathcal{F}}$ and let $L_Q := \pi(L_{\widehat{M}})$. Then*

(i) *The right* $O(q)$*-action sends leaves of* $\widehat{\mathcal{F}}$ *to leaves. Hence, the isotropy subgroup* $O(q)_{L_{\widehat{M}}} := \{A \in O(q) \mid L_{\widehat{M}} \cdot A = L_{\widehat{M}}\}$ *acts on* $L_{\widehat{M}}$.

(ii) *The restricted map* $\pi|_{L_{\widehat{M}}} : L_{\widehat{M}} \longrightarrow L_Q$ *is a covering projection.*

(iii) *Consequently,* $L_{\widehat{M}}/O(q)_{L_{\widehat{M}}}$ *can be identified with* L_Q.

Proof. (i) follows from [12, Def. 4.2.2(i)]: the right $O(q)$-action preserves $\widehat{\mathcal{F}}$, hence maps leaves to leaves.

(ii) is [12, Def. 4.2.2(ii)].

For (iii), if $x \in L_{\widehat{M}}$ and $A \in O(q)_{L_{\widehat{M}}}$, then $\pi(x \cdot A) = \pi(x)$, so $\pi|_{L_{\widehat{M}}}$ factors through $L_{\widehat{M}}/O(q)_{L_{\widehat{M}}}$. Conversely, if $x, x' \in L_{\widehat{M}}$ satisfy $\pi(x) = \pi(x')$, then $x' = x \cdot A$ for a unique $A \in O(q)$. Since the right $O(q)$-action sends leaves to leaves and $L_{\widehat{M}} \cdot A$ contains x', we have $L_{\widehat{M}} \cdot A = L_{\widehat{M}}$, so $A \in O(q)_{L_{\widehat{M}}}$. Hence $L_{\widehat{M}}/O(q)_{L_{\widehat{M}}}$ can be identified with L_Q. □

3.3 The basic Lie algebroid and the regular Molino's theory

3.3.1 Projectable transverse fields and the basic Lie algebroid

We next review projectable transverse vector fields and the associated structural Lie algebra. We use them to describe the Lie foliation structure on the fibres.

Definition 3.35 ([12, Sec. 4.1.2])**.** Let $(M, \mathcal{F})$ be a foliated manifold. Let $\mathfrak{X}(\mathcal{F}) \subseteq \mathfrak{X}(M)$ be the Lie algebra of vector fields tangent to $\mathcal{F}$. A vector field $X \in \mathfrak{X}(M)$ is *projectable* if $[X, Y] \in \mathfrak{X}(\mathcal{F})$ for all $Y \in \mathfrak{X}(\mathcal{F})$. Let $L(M, \mathcal{F})$ be the Lie algebra of projectable vector fields, and let $l(M, \mathcal{F}) := L(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$.

Definition 3.36 ([12, Secs. 4.1.1–4.1.2])**.** Let $(M, \mathcal{F})$ be a foliated manifold.

An *automorphism* of $(M, \mathcal{F})$ is a diffeomorphism $\phi : M \to M$ which preserves the foliation, equivalently, $d\phi_x(T_x\mathcal{F}) = T_{\phi(x)}\mathcal{F}$ for every $x \in M$. The group of automorphisms is denoted by $\mathrm{Aut}(M, \mathcal{F})$. The foliation $\mathcal{F}$ is called *homogeneous* if $\mathrm{Aut}(M, \mathcal{F})$ acts transitively on M.

If $\mathcal{F}$ has codimension q, then $(M, \mathcal{F})$ is called *transversely parallelizable* if there exist transverse vector fields $\bar{Y}_1, \dots, \bar{Y}_q \in l(M, \mathcal{F})$ which form a global frame of the normal bundle $N(\mathcal{F}) = TM/T\mathcal{F}$. Such a frame is called a *transverse parallelism*.

For a homogeneous foliation, let $\mathfrak{X}_{\mathrm{bas}}(\mathcal{F}) := \{X \in \mathfrak{X}(M) \mid X(f) = 0 \text{ for all } f \in \Omega^0_{\mathrm{bas}}(M, \mathcal{F})\}$. For each $x \in M$, put $E_x := \{X_x \mid X \in \mathfrak{X}_{\mathrm{bas}}(\mathcal{F})\}$, and let $E := \bigcup_{x \in M} E_x \subseteq TM$. Since $\mathcal{F}$ is homogeneous, E is an involutive subbundle of TM. The associated foliation $\mathcal{F}_{\mathrm{bas}}$, characterized by $T\mathcal{F}_{\mathrm{bas}} = E$, is called the *basic foliation* associated to $\mathcal{F}$. Since $T\mathcal{F} \subseteq T\mathcal{F}_{\mathrm{bas}}$, every leaf of $\mathcal{F}$ is contained in a leaf of $\mathcal{F}_{\mathrm{bas}}$; the leaves of $\mathcal{F}_{\mathrm{bas}}$ are called the *basic leaves*.

Proposition 3.37 (cf. [12, Sec. 6.4, Lemma 6.9])**.** *Let $\mathcal{F}$ be a transversely parallelizable foliation of codimension r on a compact manifold X. Then $W := X/\mathcal{F}_{\mathrm{bas}}$ is a smooth Hausdorff manifold, $\pi_{\mathrm{bas}} : X \to W$ is a fibre bundle, and there exists a transitive Lie algebroid $A := b(X, \mathcal{F}) \to W$ such that $\Gamma(A) \cong l(X, \mathcal{F})$ as a $C^\infty(W)$-module. Its anchor $\rho_A : A \to TW$ is induced by the $C^\infty(W)$-linear Lie algebra homomorphism $\varrho : l(X, \mathcal{F}) \longrightarrow l(X, \mathcal{F}_{\mathrm{bas}}) \cong \mathfrak{X}(W)$, and the resulting Lie algebroid A is independent, up to Lie algebroid isomorphism over W, of the choice of transverse parallelism used to identify $l(X, \mathcal{F})$ with $\Gamma(W \times \mathbb{R}^r)$.*

Proof. 1. Geometry of basic fibre bundle.

First assume that X is connected. Because $(X, \mathcal{F})$ is compact and transversely parallelizable, by [12, Theorem 4.8], it is homogeneous. Therefore [12, Theorem 4.3] applies, its basic foliation is simple and defines a smooth Hausdorff basic manifold $W = X/\mathcal{F}_{\text{bas}}$ with a fibre bundle $\pi_{\text{bas}} : X \to W$.

Now choose a transverse parallelism $\bar{Y}_1, \ldots, \bar{Y}_r \in l(X, \mathcal{F})$ where $r = \operatorname{codim} \mathcal{F}$ with representatives $Y_1, \ldots, Y_r \in L(X, \mathcal{F})$.

2. $l(X, \mathcal{F})$ is a free $C^\infty(W)$-module of rank r.

Because $\bar{Y}_1, \ldots, \bar{Y}_r$ form a global frame of the normal bundle $N(\mathcal{F})$, every $\bar{Y} \in l(X, \mathcal{F})$ can be written uniquely as $\bar{Y} = \sum_{i=1}^r a_i \bar{Y}_i$ for $a_i \in C^\infty(X)$.

Now we want to show that the a_i are basic. Let $T \in \mathfrak{X}(\mathcal{F})$. Since Y and each Y_i are projectable, $[T, Y]$ and $[T, Y_i]$ are tangent to $\mathcal{F}$. Then,

$$\begin{aligned} 0 = \overline{[T, Y]} = \overline{\left[T, \sum_{i=1}^r a_i Y_i\right]} &= \overline{\left(\sum_{i=1}^r T(a_i) Y_i + \sum_{i=1}^r a_i [T, Y_i]\right)} \\ &= \sum_{i=1}^r T(a_i) \bar{Y}_i + \sum_{i=1}^r a_i \overline{[T, Y_i]} \\ &= \sum_{i=1}^r T(a_i) \bar{Y}_i. \end{aligned}$$

Because the $\bar{Y}_i$ are pointwise linearly independent in $N(\mathcal{F})$, $T(a_i) = 0$ for all i. Therefore each a_i is basic, so $a_i = f_i \circ \pi_{\text{bas}}$ for unique $f_i \in C^\infty(W)$. Thus $\bar{Y} = \sum_{i=1}^r f_i \bar{Y}_i$ with $f_i \in C^\infty(W)$.

3. Vector bundle identifications.

After choosing $\bar{Y}_1, \ldots, \bar{Y}_r$, we define $A := W \times \mathbb{R}^r$. Let $e_1, \ldots, e_r$ be the global frame of A, and we identify sections

$$\Gamma(A) \ni \sum f_i e_i \longleftrightarrow \sum f_i \bar{Y}_i \in l(X, \mathcal{F}).$$

4. Anchor map identification.

Define

$$\varrho : l(X, \mathcal{F}) \longrightarrow l(X, \mathcal{F}_{\mathrm{bas}}) \cong \mathfrak{X}(W)$$

by projection along π_{bas} as follows. By [12, Lemma 4.5], $L(X, \mathcal{F}) \subseteq L(X, \mathcal{F}_{\mathrm{bas}})$. If $\bar{Y} \in l(X, \mathcal{F})$ is represented by $Y \in L(X, \mathcal{F})$, then Y is projectable for $\mathcal{F}_{\mathrm{bas}}$. Since the leaves of $\mathcal{F}_{\mathrm{bas}}$ are the fibres of $\pi_{\mathrm{bas}} : X \to W$, there is a unique vector field $Y_W \in \mathfrak{X}(W)$ such that $d\pi_{\mathrm{bas}}(Y_x) = (Y_W)_{\pi_{\mathrm{bas}}(x)}$ for every $x \in X$. Let $\varrho(\bar{Y}) := Y_W$, using the identification $l(X, \mathcal{F}_{\mathrm{bas}}) \cong \mathfrak{X}(W)$.

We check that ϱ is well-defined. If $Y' = Y + U$ with $U \in \mathfrak{X}(\mathcal{F})$, then U is tangent to $\mathcal{F}_{\mathrm{bas}}$ as well, so $d\pi_{\mathrm{bas}}(U) = 0$. Hence Y and Y' project to the same vector field on W.

Then we check that ϱ is $C^\infty(W)$-linear and a Lie algebra homomorphism: For $f \in C^\infty(W)$, the section $f\bar{Y}$ is represented by $(f \circ \pi_{\mathrm{bas}})Y$. If Y projects to Y_W, then $(f \circ \pi_{\mathrm{bas}})Y$ projects to fY_W, so $\varrho(f\bar{Y}) = f\varrho(\bar{Y})$. If $Y, Y' \in L(X, \mathcal{F})$ project to $Y_W, Y'_W \in \mathfrak{X}(W)$, then $[Y, Y']$ projects to $[Y_W, Y'_W]$. Hence $\varrho([\bar{Y}, \bar{Y}']) = [\varrho(\bar{Y}), \varrho(\bar{Y}')]$.

Under the identification $\Gamma(A) \cong l(X, \mathcal{F})$, the map ϱ therefore induces a bundle map

$$\rho_A : A \to TW,$$

which is the anchor.

5. Define bracket.

We define the bracket on $\Gamma(A)$ by transporting the Lie bracket on $l(X,\mathcal{F})$ through the identification

$$\Phi : \Gamma(A) \xrightarrow{\cong} l(X,\mathcal{F}), \qquad \sum_{i=1}^{r} f_i e_i \longmapsto \sum_{i=1}^{r} f_i \bar{Y}_i.$$

Thus, for $\xi, \eta \in \Gamma(A)$, we write $[\xi,\eta]_A := \Phi^{-1}\big([\Phi(\xi), \Phi(\eta)]\big)$.

Equivalently, if $\bar{Y}, \bar{Z} \in l(X,\mathcal{F})$ are represented by $Y, Z \in L(X,\mathcal{F})$, then $[\bar{Y},\bar{Z}] := \overline{[Y,Z]}$.

This is well defined: if $Y' = Y + U$ and $Z' = Z + V$ with $U, V \in \mathfrak{X}(\mathcal{F})$, then $[Y', Z'] - [Y, Z] = [U, Z] + [Y, V] + [U, V] \in \mathfrak{X}(\mathcal{F})$. Skew-symmetry and the Jacobi identity follow from the corresponding properties of the Lie bracket on $l(X,\mathcal{F})$.

It remains to verify the Leibniz rule. Let $\xi, \eta \in \Gamma(A)$ correspond to $\bar{Y}, \bar{Z} \in l(X,\mathcal{F})$, and let $f \in C^\infty(W)$. If $Y, Z \in L(X,\mathcal{F})$ represent $\bar{Y}, \bar{Z}$, then $(f \circ \pi_{\mathrm{bas}})Z$ represents $f\eta$. Hence

$$[\xi, f\eta]_A = \Phi^{-1}\big(\overline{[Y, (f \circ \pi_{\mathrm{bas}})Z]}\big).$$

Using the Leibniz rule for vector fields on X, we get

$$[Y, (f \circ \pi_{\mathrm{bas}})Z] = (f \circ \pi_{\mathrm{bas}})[Y, Z] + Y(f \circ \pi_{\mathrm{bas}})\, Z.$$

Since $\rho_A(\xi) = \varrho(\bar{Y})$ is the vector field on W projected from Y, we have

$$Y(f \circ \pi_{\mathrm{bas}}) = (\rho_A(\xi) f) \circ \pi_{\mathrm{bas}}.$$

Therefore

$$[\xi, f\eta]_A = f[\xi,\eta]_A + \rho_A(\xi)(f)\eta.$$

Since ρ_A is induced by the $C^\infty(W)$-linear Lie algebra homomorphism $\varrho : l(X, \mathcal{F}) \to \mathfrak{X}(W)$, the anchor preserves brackets. Hence $(A, [\,,\,]_A, \rho_A)$ is a Lie algebroid over W.

6. A is transitive.

By definition, this means that the anchor is surjective at every point [12, Lemma 6.9].

Let $w \in W$ and $v \in T_w W$. Let $x \in X$ with $\pi_{\text{bas}}(x) = w$. Since π_{bas} is a submersion, there exists $\zeta \in T_x X$ such that $d\pi_{\text{bas}}(\zeta) = v$. Recall that we chose representatives $Y_1, \dots, Y_r \in L(X, \mathcal{F})$. Then $(Y_1)_x, \dots, (Y_r)_x$ span a subspace complementary to $T_x\mathcal{F}$ in $T_x X$. Therefore, there exist scalars $a_1, \dots, a_r \in \mathbb{R}$ such that

$$a_1(Y_1)_x + \cdots + a_r(Y_r)_x - \zeta \in T_x\mathcal{F}.$$

Let $Y := a_1 Y_1 + \cdots + a_r Y_r$. Then $Y \in L(X, \mathcal{F})$. Since $T_x\mathcal{F} \subseteq \ker(d\pi_{\text{bas}})_x$, we get $d\pi_{\text{bas}}(Y_x) = d\pi_{\text{bas}}(\zeta) = v$. Therefore the class $\bar{Y} \in l(X, \mathcal{F}) \cong \Gamma(A)$ satisfies $\rho_A(\bar{Y})(w) = v$. Since v was arbitrary, $\rho_{A,w}$ is surjective. Since w was arbitrary as well, ρ_A is surjective everywhere. So A is transitive.

We also get an exact sequence

$$0 \longrightarrow \ker \rho_A \longrightarrow A \xrightarrow{\rho_A} TW \longrightarrow 0.$$

This is the transitive basic Lie algebroid attached to $(X, \mathcal{F})$.

7. A is independent of the transverse parallelism.

Suppose $\bar{Y}_1, \ldots, \bar{Y}_r$ and $\bar{Y}_1', \ldots, \bar{Y}_r'$ are two transverse parallelisms of $(X, \mathcal{F})$. They yield identifications

$$\Phi, \Phi' : \Gamma(W \times \mathbb{R}^r) \xrightarrow{\cong} l(X, \mathcal{F}),$$

hence two Lie algebroid structures $([\,,\,], \rho)$ and $([\,,\,]', \rho')$ on $W \times \mathbb{R}^r$.

Let

$$T := (\Phi')^{-1} \circ \Phi : \Gamma(W \times \mathbb{R}^r) \to \Gamma(W \times \mathbb{R}^r).$$

Then for any sections ξ, η,

$$T([\xi, \eta]) = (\Phi')^{-1}\Phi\big(\Phi^{-1}([\Phi(\xi), \Phi(\eta)])\big) = (\Phi')^{-1}\big([\Phi'(T\xi), \Phi'(T\eta)]\big) = [T\xi, T\eta]'.$$

Also,

$$\rho'(T\xi) = \varrho(\Phi'(T\xi)) = \varrho(\Phi(\xi)) = \rho(\xi).$$

Hence T preserves both the bracket and the anchor, so it is an isomorphism of Lie algebroids. Therefore the Lie algebroid A is independent, up to isomorphism, of the chosen transverse parallelism.

8. Disconnected case.

Let $X = \bigsqcup_\alpha X_\alpha$ be the decomposition into connected components and $\mathcal{F}_\alpha := \mathcal{F}|_{X_\alpha}$. Applying the discussion in the connected case to each $(X_\alpha, \mathcal{F}_\alpha)$, we obtain W_α, $\pi_{\mathrm{bas},\alpha} : X_\alpha \to W_\alpha$, and a transitive Lie algebroid $A_\alpha \to W_\alpha$ such that $\Gamma(A_\alpha) \cong l(X_\alpha, \mathcal{F}_\alpha)$.

Let $W := \bigsqcup_\alpha W_\alpha$, $\pi_{\mathrm{bas}} := \bigsqcup_\alpha \pi_{\mathrm{bas},\alpha} : X \to W$, and $A := \bigsqcup_\alpha A_\alpha \to W$.

Since all structures are defined componentwise, the anchor and bracket on A are obtained by taking the disjoint union of the anchors and brackets on A_α, so A is a transitive Lie algebroid over W. A section of A is a family of sections of A_α, and likewise a transverse projectable field on $(X,\mathcal{F})$ is a family of transverse projectable fields on $(X_\alpha,\mathcal{F}_\alpha)$. Hence $\Gamma(A)\cong l(X,\mathcal{F})$ as $C^\infty(W)$-modules. This proves the disconnected case. □

3.3.2 Structural Lie algebras on Molino fibres

Lemma 3.38. *Let $\mathcal{F}$ be a transversely parallelizable foliation of a compact manifold X, let $\pi_{\mathrm{bas}}: X\to W := X/\mathcal{F}_{\mathrm{bas}}$ be the basic fibre bundle, and let $A := b(X,\mathcal{F})\to W$ be the basic Lie algebroid. For $w\in W$, write $L_w := \pi_{\mathrm{bas}}^{-1}(w)$. Then restriction to the fibre L_w induces a Lie algebra isomorphism*

$$(\ker\rho_A)_w \cong l(L_w,\mathcal{F}|_{L_w}),$$

and $(L_w,\mathcal{F}|_{L_w})$ is a Lie foliation.

Proof. Let $w\in W$ and $L_w=\pi_{\mathrm{bas}}^{-1}(w)$. By Proposition 3.37, we have that $A=b(X,\mathcal{F})\to W$ is a transitive Lie algebroid with $\Gamma(A)\cong l(X,\mathcal{F})$ as a $C^\infty(W)$-module, and its anchor ρ_A is induced by the $C^\infty(W)$-linear Lie algebra homomorphism $\varrho: l(X,\mathcal{F})\longrightarrow l(X,\mathcal{F}_{\mathrm{bas}})\cong\mathfrak{X}(W)$.

Step 1: We want to rewrite the isotropy fibre as a quotient.

Let

$$I_w := \{f\in C^\infty(W)\mid f(w)=0\},\qquad \mathfrak{k}_w := \{\bar{Y}\in l(X,\mathcal{F})\mid \varrho(\bar{Y})(w)=0\}.$$

Since $\Gamma(A)\cong l(X,\mathcal{F})$, the identification $A_w\cong\Gamma(A)/I_w\Gamma(A)$ gives $A_w\cong l(X,\mathcal{F})/I_w\, l(X,\mathcal{F})$. Under this identification, $\rho_{A,w}: A_w\to T_wW$ is induced by $\bar{Y}\mapsto\varrho(\bar{Y})(w)$. Therefore, $(\ker\rho_A)_w\cong\mathfrak{k}_w/I_w\, l(X,\mathcal{F})$.

Remark: At this point we want to discuss that $\mathfrak{k}_w/I_w\, l(X,\mathcal{F})$ carries the correct Lie algebra structure.

1. $I_w\, l(X,\mathcal{F}) \subseteq \mathfrak{k}_w$: if $f \in I_w$ and $\bar{Y} \in l(X,\mathcal{F})$, then $\varrho(f\bar{Y})(w) = f(w)\varrho(\bar{Y})(w) = 0$.

2. $\mathfrak{k}_w$ is a Lie subalgebra of $l(X,\mathcal{F})$: let $\mathfrak{m}_w := \{Z \in \mathfrak{X}(W) : Z_w = 0\}$. It is easy to show that $\mathfrak{m}_w$ is a Lie subalgebra of $\mathfrak{X}(W)$. Because ϱ is a Lie algebra homomorphism, $\mathfrak{k}_w = \varrho^{-1}(\mathfrak{m}_w)$ is a Lie subalgebra of $l(X,\mathcal{F})$.

3. $I_w\, l(X,\mathcal{F})$ is an ideal in $\mathfrak{k}_w$: by the Leibniz rule, if $\bar{Y} \in \mathfrak{k}_w$ and $f\bar{Z} \in I_w\, l(X,\mathcal{F})$, then $[\bar{Y}, f\bar{Z}] = f[\bar{Y},\bar{Z}] + (\varrho(\bar{Y})f)\bar{Z} \in I_w\, l(X,\mathcal{F})$, because $(\varrho(\bar{Y})f)(w) = df_w(\varrho(\bar{Y})(w)) = 0$.

Step 2: We construct the restriction map to the fibre.

Let $\bar{Y} \in \mathfrak{k}_w$, choose a representative $Y \in L(X,\mathcal{F})$, and let $Z := \varrho(\bar{Y}) \in \mathfrak{X}(W)$. Since $\bar{Y} \in \mathfrak{k}_w$, we have $Z_w = 0$, so for every $x \in L_w$ we have

$$d\pi_{\mathrm{bas}}(Y_x) = Z_{\pi_{\mathrm{bas}}(x)} = Z_w = 0.$$

Hence $Y_x \in \ker(d\pi_{\mathrm{bas}})_x = T_xL_w$, so Y restricts to a vector field $Y|_{L_w} \in \mathfrak{X}(L_w)$.

To check that $Y|_{L_w}$ is projectable for $\mathcal{F}|_{L_w}$, let $X_0 \in \mathfrak{X}(\mathcal{F}|_{L_w})$. Choose a local extension $\widetilde{X}_0 \in \mathfrak{X}(\mathcal{F})$ near each point of L_w with $\widetilde{X}_0|_{L_w} = X_0$. Since $Y \in L(X,\mathcal{F})$, we have $[Y,\widetilde{X}_0] \in \mathfrak{X}(\mathcal{F})$, hence also $[\widetilde{X}_0, Y] \in \mathfrak{X}(\mathcal{F})$. Thus for every $x \in L_w$, $[\widetilde{X}_0, Y](x) \in T_x\mathcal{F} \subseteq T_x\mathcal{F}_{\mathrm{bas}} = T_xL_w$. This implies that $[\widetilde{X}_0, Y]|_{L_w}$ is a well-defined vector field on L_w. If $f \in C^\infty(L_w)$ and $\widetilde{f}$ is an extension of f, then

$$([\widetilde{X}_0, Y]|_{L_w})(f) = [\widetilde{X}_0, Y](\widetilde{f})|_{L_w}.$$

We compute that

$$\begin{aligned}
[\widetilde{X}_0, Y](\widetilde{f})|_{L_w} &= \widetilde{X}_0(Y\widetilde{f})|_{L_w} - Y(\widetilde{X}_0\widetilde{f})|_{L_w} \\
&= X_0((Y\widetilde{f})|_{L_w}) - Y|_{L_w}((\widetilde{X}_0\widetilde{f})|_{L_w}) \\
&= X_0(Y|_{L_w}(f)) - Y|_{L_w}(X_0(f)) \\
&= [X_0, Y|_{L_w}](f).
\end{aligned}$$

This is to say $[X_0, Y|_{L_w}] = [\widetilde{X}_0, Y]|_{L_w} \in \mathfrak{X}(\mathcal{F}|_{L_w})$, so $Y|_{L_w} \in L(L_w, \mathcal{F}|_{L_w})$, and we may define

$$\mathrm{Res}_w(\bar{Y}) := \overline{Y|_{L_w}} \in l(L_w, \mathcal{F}|_{L_w}).$$

This is well defined: if Y' is another representative of $\bar{Y}$, then $Y' - Y \in \mathfrak{X}(\mathcal{F})$, so $(Y' - Y)|_{L_w} \in \mathfrak{X}(\mathcal{F}|_{L_w})$, hence $\overline{Y'|_{L_w}} = \overline{Y|_{L_w}}$. It is a Lie algebra homomorphism because restriction commutes with Lie brackets for vector fields tangent to L_w, so $\mathrm{Res}_w([\bar{Y}, \bar{Z}]) = [\mathrm{Res}_w(\bar{Y}), \mathrm{Res}_w(\bar{Z})]$.

Now let $f \in I_w$ and let $\bar{Y} \in l(X, \mathcal{F})$ be represented by $Y \in L(X, \mathcal{F})$. Since $\pi_{\mathrm{bas}}|_{L_w} \equiv w$ and $f\bar{Y} \in I_w\, l(X, \mathcal{F}) \subseteq \mathfrak{k}_w$, we have

$$\mathrm{Res}_w(f\bar{Y}) = \overline{((f \circ \pi_{\mathrm{bas}})Y)|_{L_w}} = \overline{f(w)Y|_{L_w}} = 0.$$

Therefore Res_w vanishes on $I_w\, l(X, \mathcal{F})$ and hence descends to a Lie algebra homomorphism

$$\mathrm{res}_w : (\ker \rho_A)_w \longrightarrow l(L_w, \mathcal{F}|_{L_w}).$$

Step 3: It remains to prove that res_w is an isomorphism.

Let $x \in L_w$. Choose a transverse parallelism $\bar{Y}_1, \ldots, \bar{Y}_r$ on $(X, \mathcal{F})$. Then $l(X, \mathcal{F})$ is a free $C^\infty(W)$-module with basis $\bar{Y}_1, \ldots, \bar{Y}_r$, so the classes $[\bar{Y}_1]_w, \ldots, [\bar{Y}_r]_w$ in $A_w \cong l(X, \mathcal{F})/I_w\, l(X, \mathcal{F})$ form a basis of A_w. Since $(\bar{Y}_1)_x, \ldots, (\bar{Y}_r)_x$ form a basis of $N_x(\mathcal{F})$, evaluation at x defines a linear isomorphism

$$\mathrm{ev}_x^X : A_w \longrightarrow N_x(\mathcal{F}), \qquad [\bar{Y}] \longmapsto \bar{Y}_x.$$

This map is well defined because if $\bar{s} = f\bar{Y} \in I_w\, l(X, \mathcal{F})$, then for $x \in L_w$, $(f \circ \pi_{\mathrm{bas}})(x) = f(w) = 0$, so

$$\bar{s}_x = \overline{((f \circ \pi_{\mathrm{bas}})Y)_x} = 0.$$

Similarly, by [12, Theorem 4.9], $(L_w, \mathcal{F}|_{L_w})$ is transversely parallelizable and all its basic functions are constant, so the same argument as in the proof of [12, Theorem 4.24] shows that the evaluation map

$$\mathrm{ev}_x^L : l(L_w, \mathcal{F}|_{L_w}) \longrightarrow N_x(\mathcal{F}|_{L_w}), \qquad \bar{Z} \longmapsto \bar{Z}_x,$$

is a linear isomorphism.

Under the isomorphism ev_x^X, we claim that the anchor $\rho_{A,w} : A_w \to T_wW$ corresponds to the map induced by $d\pi_{\mathrm{bas}}$ on normal spaces, $\overline{d\pi_{\mathrm{bas}}} : N_x(\mathcal{F}) \to T_wW$, $[v] \mapsto d\pi_{\mathrm{bas}}(v)$.

Indeed, if $\bar{Y} \in l(X, \mathcal{F})$ is represented by $Y \in L(X, \mathcal{F})$, then $d\pi_{\mathrm{bas}}(Y) = \varrho(\bar{Y}) \circ \pi_{\mathrm{bas}}$, and therefore $\overline{d\pi_{\mathrm{bas}}}(\bar{Y}_x) = \varrho(\bar{Y})(w) = \rho_{A,w}([\bar{Y}])$. Hence ev_x^X restricts to an isomorphism

$$(\ker \rho_A)_w \xrightarrow{\cong} \ker\big(\overline{d\pi_{\mathrm{bas}}} : N_x(\mathcal{F}) \to T_wW\big) = T_xL_w/T_x(\mathcal{F}) = N_x(\mathcal{F}|_{L_w}),$$

since $T_xL_w = \ker(d\pi_{\mathrm{bas}})_x$ and $T_x(\mathcal{F}|_{L_w}) = T_x(\mathcal{F})$.

Finally, for $\xi = [\bar{Y}] \in (\ker \rho_A)_w$, we have

$$\mathrm{ev}^L_x(\mathrm{res}_w(\xi)) = \mathrm{ev}^L_x(\overline{Y|_{L_w}}) = \overline{Y_x} = \mathrm{ev}^X_x(\xi).$$

This is saying that the following square commutes:

$$\begin{array}{ccc} (\ker \rho_A)_w & \xrightarrow{\mathrm{res}_w} & l(L_w, \mathcal{F}|_{L_w}) \\ {\scriptstyle \mathrm{ev}^X_x|_{(\ker \rho_A)_w}} \downarrow \cong & & {\scriptstyle \mathrm{ev}^L_x} \downarrow \cong \\ N_x(\mathcal{F}|_{L_w}) & = & N_x(\mathcal{F}|_{L_w}) \end{array}$$

Thus $\mathrm{ev}^L_x \circ \mathrm{res}_w = \mathrm{ev}^X_x \big|_{(\ker \rho_A)_w}$. Both maps are isomorphisms onto $N_x(\mathcal{F}|_{L_w})$, so res_w is an isomorphism. Since it is already a Lie algebra homomorphism, it is a Lie algebra isomorphism.

Step 4: We want to show that $(L_w, \mathcal{F}|_{L_w})$ is a Lie foliation.

Since $L_w = \pi^{-1}_{\mathrm{bas}}(w)$ and X is compact, the fibre L_w is compact. Since L_w is a leaf of $\mathcal{F}_{\mathrm{bas}}$, it is connected. By [12, Theorem 4.9], the foliated manifold $(L_w, \mathcal{F}|_{L_w})$ is transversely parallelizable and its basic functions are constant. Hence [12, Theorem 4.24] implies that $(L_w, \mathcal{F}|_{L_w})$ is a Lie foliation. □

Proposition 3.39. *For every $w_0 \in W$, there exists an open neighborhood $U \subseteq W$ of w_0 and sections $\sigma_1, \ldots, \sigma_s \in \Gamma(U, \ker \rho_A)$, where $s := \mathrm{rank}(\ker \rho_A)$, such that, for every $w \in U$, the elements $\mathrm{res}_w(\sigma_1(w)), \ldots, \mathrm{res}_w(\sigma_s(w))$ form a basis of $l(L_w, \mathcal{F}|_{L_w})$.*

Proof. Since $A \to W$ is transitive, $\rho_A : A \to TW$ is surjective at every point and has constant rank $\dim W$. Hence $\ker \rho_A \subseteq A$ is a smooth vector subbundle.

Shrink around w_0 and choose a smooth local frame $\sigma_1, \ldots, \sigma_s$ of $\ker \rho_A|_U$. We can do this because there is a vector bundle isomorphism $\Phi : (\ker \rho_A)|_U \xrightarrow{\cong} U \times \mathbb{R}^s$. Let $\varepsilon_1, \ldots, \varepsilon_s$ be the standard basis of $\mathbb{R}^s$. Using Φ^{-1}, these sections are given by

$$\sigma_p(w) := \Phi^{-1}(w, \varepsilon_p), \qquad p = 1, \ldots, s.$$

Each σ_p is smooth because Φ^{-1} is smooth.

Choose a transverse parallelism on $(X, \mathcal{F})$, and let $e_1, \ldots, e_r$ be the corresponding global frame of A, where $r = \operatorname{rank}(A) = \operatorname{codim} \mathcal{F}$. Let $Y_1, \ldots, Y_r \in L(X, \mathcal{F})$ be projectable vector fields representing $e_1, \ldots, e_r$, respectively.

On U, let

$$\sigma_p = \sum_{i=1}^{r} c_{pi} e_i, \quad c_{pi} \in C^\infty(U), \quad p = 1, \ldots, s.$$

Define on $\pi_{\text{bas}}^{-1}(U)$,

$$\widetilde{Y}_p := \sum_{i=1}^{r} (c_{pi} \circ \pi_{\text{bas}}) Y_i.$$

We construct $\widetilde{Y}_p$ this way because the $C^\infty(W)$-module structure on $\Gamma(A) \cong l(X, \mathcal{F})$ is implemented upstairs by pullback along π_{bas}: $f \cdot \bar{Y} \leftrightarrow (f \circ \pi_{\text{bas}}) Y$. Because $c_{pi} \circ \pi_{\text{bas}}$ are basic and Y_i are foliate, each $\widetilde{Y}_p$ belongs to $L(\pi_{\text{bas}}^{-1}(U), \mathcal{F})$. By construction, $\widetilde{Y}_p$ represents the section σ_p.

Since $\sigma_p \in \Gamma(U, \ker \rho_A)$, its anchor vanishes, so the projected vector field of $\widetilde{Y}_p$ on U is zero. Equivalently, $d\pi_{\text{bas}}(\widetilde{Y}_p) = 0$, so for each $x \in L_w$ we have

$$\widetilde{Y}_p(x) \in \ker(d\pi_{\text{bas}})_x = T_x L_w.$$

Hence for each $w \in U$, $\widetilde{Y}_p|_{L_w}$ is a vector field on L_w. Since $\widetilde{Y}_p$ is foliated for $\mathcal{F}$, its restriction is foliate for $\mathcal{F}|_{L_w}$. This is similar to what we did in Lemma 3.38.

By construction, $\mathrm{res}_w(\sigma_p(w)) = \overline{\widetilde{Y}_p|_{L_w}}$. Because $\sigma_1, \ldots, \sigma_s$ is a local frame of $\ker \rho_A|_U$, the vectors $\sigma_1(w), \ldots, \sigma_s(w)$ form a basis of $(\ker \rho_A)_w$ for each $w \in U$. Then, by Lemma 3.38, the elements $\mathrm{res}_w(\sigma_1(w)), \ldots, \mathrm{res}_w(\sigma_s(w))$ form a basis of $l(L_w, \mathcal{F}|_{L_w})$. □

Remark 3.40. Consider the local triviality of the fibre bundle π_{bas} after shrinking U if necessary. We choose $\Phi : \pi_{\mathrm{bas}}^{-1}(U) \xrightarrow{\cong} U \times L$, where L is diffeomorphic to the fibre over w_0. Under Φ, the vector fields $\widetilde{Y}_1, \ldots, \widetilde{Y}_s$ become smooth vector fields tangent to the slices $\{w\} \times L$. For each $w \in U$, their classes on the slice $\{w\} \times L$ form a basis of the corresponding structural Lie algebra.

3.3.3 Molino's structure theorem for regular Riemannian groupoids

Theorem 3.41 (Molino's structure theorem for regular Riemannian groupoids). *Let $G \rightrightarrows M$ be a Hausdorff regular Lie groupoid with M compact and connected, and let $\eta^{(2)}$ be a 2-metric on $G^{(2)}$. Let $\mathcal{O}$ be the orbit foliation, let $N := TM/T(\mathcal{O})$, let $q := \mathrm{rank}\, N$, and let g^N be the quotient metric on N induced by the 0-metric on M determined by $\eta^{(2)}$. Let $\pi : OF(M, \mathcal{O}) \to M$ be the transverse orthonormal frame bundle of (N, g^N), and let $\widetilde{\mathcal{O}}$ be the lifted foliation.*

Then the orbit foliation $(M, \mathcal{O})$ is a Riemannian foliation. Moreover:

(i) *The foliated manifold $\big(OF(M, \mathcal{O}), \widetilde{\mathcal{O}}\big)$ is transversely parallelizable.*

(ii) *There exists a smooth manifold B with a smooth right $O(q)$-action and an $O(q)$-equivariant fibre bundle $\kappa : OF(M, \mathcal{O}) \to B$ whose fibres are the closures of the leaves of $\widetilde{\mathcal{O}}$.*

(iii) *For each* $b \in B$*, let* $L_b := \kappa^{-1}(b)$*. Then the restricted foliation* $\widetilde{\mathcal{O}}|_{L_b}$ *is a Lie foliation with dense holonomy group, and the Lie algebra* $l\big(L_b, \widetilde{\mathcal{O}}|_{L_b}\big)$ *is independent of* $b \in B$ *up to isomorphism.*

If $\pi_{\mathrm{bas}} : OF(M, \mathcal{O}) \to W$ *is the basic fibre bundle of* $\big(OF(M, \mathcal{O}), \widetilde{\mathcal{O}}\big)$*, and if* $A \to W$ *denotes the associated basic Lie algebroid, then there is a unique diffeomorphism* $\Phi : B \xrightarrow{\cong} W$ *such that* $\pi_{\mathrm{bas}} = \Phi \circ \kappa$*. Via* Φ*, we regard* A *as a transitive Lie algebroid over* B*, and restriction to the fibre induces a Lie algebra isomorphism* $(\ker \rho_A)_b \xrightarrow{\cong} l\big(L_b, \widetilde{\mathcal{O}}|_{L_b}\big)$*. Hence the structural Lie algebra of* $\widetilde{\mathcal{O}}|_{L_b}$ *is identified with* $(\ker \rho_A)_b$*.*

(iv) *The normal representation* $\lambda^N : G \curvearrowright N$ *lifts canonically to a smooth left action of* G *on* $OF(M, \mathcal{O})$ *with moment map* π*, given by* $g \cdot e = \lambda^N_g \circ e$*, and this action commutes with the right* $O(q)$*-action.*

For every local bisection $\sigma : U \to G$ *with* $s \circ \sigma = \mathrm{id}_U$ *and* $t \circ \sigma : U \to t(\sigma(U))$ *a diffeomorphism, the induced map* $f_\sigma : \pi^{-1}(U) \to \pi^{-1}(t(\sigma(U)))$*,* $f_\sigma(e) := \sigma(\pi(e)) \cdot e$*, preserves* $\widetilde{\mathcal{O}}$*,* θ*, and* ω*. It sends each* κ*-fibre contained in* $\pi^{-1}(U)$ *onto a* κ*-fibre, and hence descends to a local* $O(q)$*-equivariant diffeomorphism* $\bar{f}_\sigma$ *of* B*. These descended maps are compatible with products and inverses of local bisections on their common domains; thus the local bisections of* G *define a pseudogroup of local* $O(q)$*-equivariant diffeomorphisms of* B*, and, via the identification* $\Phi : B \xrightarrow{\cong} W$ *in* (iii)*, a pseudogroup of local diffeomorphisms of* W*.*

For each b *in the domain of* $\bar{f}_\sigma$*, the restriction* $f_\sigma|_{L_b} : L_b \to L_{\bar{f}_\sigma(b)}$ *is a foliated diffeomorphism. Hence it induces an isomorphism of structural Lie algebras, which, under the identifications in* (iii)*, gives an isomorphism* $(\ker \rho_A)_b \xrightarrow{\cong} (\ker \rho_A)_{\bar{f}_\sigma(b)}$*.*

The following discussion provides the remaining technical inputs for the proof.

3.3.4 Descent and Maurer–Cartan forms

The following records a descent lemma that will be used later, after κ is constructed, to push bisection-induced maps down to the base B.

Lemma 3.42. *Let $\kappa : OF(M, \mathcal{O}) \to B$ be a fibre bundle whose fibres are the closures of the leaves of $\widetilde{\mathcal{O}}$, and assume that κ is proper. Let $U, V \subseteq M$ be open and let $f : \pi^{-1}(U) \to \pi^{-1}(V)$ be a diffeomorphism sending leaves of $\widetilde{\mathcal{O}}$ to leaves. Let $B_U := \{b \in B \mid \kappa^{-1}(b) \subseteq \pi^{-1}(U)\}$, $B_V := \{b \in B \mid \kappa^{-1}(b) \subseteq \pi^{-1}(V)\}$. Then B_U and B_V are open.*

For every $b \in B_U$, there exists a unique $b' \in B_V$ such that $f(\kappa^{-1}(b)) = \kappa^{-1}(b')$. $b \mapsto b'$ defines a smooth diffeomorphism $\bar{f} : B_U \to B_V$ satisfying $\kappa \circ f = \bar{f} \circ \kappa$ on $\kappa^{-1}(B_U)$.

Proof. Since κ is proper, it is a closed map. Hence $B_U = B \setminus \kappa\big(OF(M, \mathcal{O}) \setminus \pi^{-1}(U)\big)$ is open, and similarly for B_V.

Let $b \in B_U$ and choose $e \in \kappa^{-1}(b)$. Let L be the leaf of $\widetilde{\mathcal{O}}$ through e. By definition of κ, $\overline{L} = \kappa^{-1}(b)$. Since $b \in B_U$, we have $\kappa^{-1}(b) \subseteq \pi^{-1}(U)$, and therefore $L \subseteq \overline{L} = \kappa^{-1}(b) \subseteq \pi^{-1}(U)$. Hence $f(L)$ is defined, and it is a leaf of $\widetilde{\mathcal{O}}$ contained in $\pi^{-1}(V)$.

Because $\kappa^{-1}(b)$ is compact and $f : \pi^{-1}(U) \to \pi^{-1}(V)$ is a homeomorphism, the image $f(\kappa^{-1}(b))$ is compact and therefore closed in $OF(M, \mathcal{O})$. Since L is dense in $\kappa^{-1}(b)$, the leaf $f(L)$ is dense in $f(\kappa^{-1}(b))$. Hence $\overline{f(L)} = f(\overline{L}) = f(\kappa^{-1}(b))$. As $f(L)$ is the leaf through $f(e)$, this gives $f(\kappa^{-1}(b)) = \overline{f(L)} = \kappa^{-1}(b')$.

We want to show that b' is independent of the choice of e. Let $e_1, e_2 \in \kappa^{-1}(b)$, and let L_{e_1} be the leaf of $\widetilde{\mathcal{O}}$ through e_1. Then $e_2 \in \overline{L_{e_1}} = \kappa^{-1}(b)$. Applying f, we have $f(e_2) \in f(\kappa^{-1}(b)) = \kappa^{-1}(b')$, hence $\kappa(f(e_2)) = \kappa(f(e_1)) = b'$. So $\bar{f}(b) = b'$ is well-defined and satisfies $\kappa \circ f = \bar{f} \circ \kappa$ on $\kappa^{-1}(B_U)$. Bijectivity then follows by applying the same construction to f^{-1}.

Finally, $\bar{f}$ is smooth because κ is a fibre bundle. For any $b \in B_U$, choose a local section $s : W \to OF(M, \mathcal{O})$ of κ with $b \in W \subseteq B_U$. Then on W we have $\bar{f} = \kappa \circ f \circ s$, hence $\bar{f}$ is smooth. The same applies to $\bar{f}^{-1}$. $\square$

Lemma 3.43. *Let $\delta : U \to G$ and $\tau : V \to G$ be local bisections such that $t(\delta(U)) \subseteq V$. For any local bisection $\sigma : W \to G$, let $f_\sigma : \pi^{-1}(W) \to \pi^{-1}(t(\sigma(W)))$ be the induced map on $OF(M, \mathcal{O})$, and let $B_W := \{b \in B \mid \kappa^{-1}(b) \subseteq \pi^{-1}(W)\}$.*

f_σ descends to a unique diffeomorphism $\bar{f}_\sigma : B_W \to B_{t(\sigma(W))}$ such that $\kappa \circ f_\sigma = \bar{f}_\sigma \circ \kappa$ on $\kappa^{-1}(B_W)$. Then:

(1) *On $\pi^{-1}(U)$, $f_{\tau*\delta} = f_\tau \circ f_\delta$, where $\tau * \delta$ denotes the product bisection.*

(2) *On B_U, $\bar{f}_{\tau*\delta} = \bar{f}_\tau \circ \bar{f}_\delta$.*

(3) *For every $b \in B_U$ and every $A \in O(q)$, $\bar{f}_\delta(b \cdot A) = \bar{f}_\delta(b) \cdot A$.*

Proof. (1) Let $e \in \pi^{-1}(U)$ and $x = \pi(e)$. Then

$$f_{\tau*\delta}(e) = (\tau * \delta)(x) \cdot e = \big(\tau(t(\delta(x)))\delta(x)\big) \cdot e = \tau(t(\delta(x))) \cdot \big(\delta(x) \cdot e\big) = f_\tau\big(f_\delta(e)\big),$$

where we used that $\pi(\delta(x) \cdot e) = t(\delta(x))$ and $t(\delta(U)) \subseteq V$.

(2) Since $t(\delta(U)) \subseteq V$, we have $\bar{f}_\delta(B_U) \subseteq B_V$, so $\bar{f}_\tau \circ \bar{f}_\delta$ is defined on B_U. For $e \in \kappa^{-1}(B_U) \subseteq \pi^{-1}(U)$,

$$\kappa\big(f_{\tau*\delta}(e)\big) = \kappa\big(f_\tau(f_\delta(e))\big) = \bar{f}_\tau\big(\kappa(f_\delta(e))\big) = \bar{f}_\tau\big(\bar{f}_\delta(\kappa(e))\big) = (\bar{f}_\tau \circ \bar{f}_\delta)\big(\kappa(e)\big).$$

By uniqueness in Lemma 3.42, this yields $\bar{f}_{\tau*\delta} = \bar{f}_\tau \circ \bar{f}_\delta$ on B_U.

(3) Since f_δ is $O(q)$-equivariant and κ is $O(q)$-equivariant, for $e \in \kappa^{-1}(b)$ we have

$$\bar{f}_\delta(b \cdot A) = \kappa\big(f_\delta(e \cdot A)\big) = \kappa\big(f_\delta(e) \cdot A\big) = \kappa\big(f_\delta(e)\big) \cdot A = \bar{f}_\delta(b) \cdot A.$$

□

Lemma 3.44. *Let X be a manifold, and let $p : X \to B$, $q : X \to W$ be surjective submersions onto Hausdorff manifolds. Assume that p and q have the same fibres. Then there exists a unique diffeomorphism $\Phi : B \xrightarrow{\cong} W$ such that $q = \Phi \circ p$.*

Proof. Since q is constant on the fibres of p, the quotient property of p gives a unique continuous map $\Phi : B \to W$ with $q = \Phi \circ p$. Similarly, there is a unique continuous map $\Psi : W \to B$ with $p = \Psi \circ q$. Then it is straightforward to check that $(\Psi \circ \Phi) \circ p = \Psi \circ q = p$, so $\Psi \circ \Phi = \mathrm{id}_B$ because p is surjective. Likewise, $\Phi \circ \Psi = \mathrm{id}_W$. Thus Φ is a homeomorphism.

To prove smoothness, let $b \in B$. Since p is a submersion, there is a local section $s : U \to X$ of p through b. On U, $\Phi|_U = q \circ s$, so Φ is smooth. The same argument applied to Ψ shows that Φ^{-1} is smooth. Hence Φ is a diffeomorphism. □

Definition 3.45 ([12, Sec. 4.3.1])**.** Recall that a $\mathfrak{g}$-valued 1-form η is a Maurer–Cartan form if $d\eta + \frac{1}{2}[\eta, \eta] = 0$.

Remark 3.46 ([12, Theorem 4.24])*.* If $(L, \mathcal{F})$ is transversely parallelizable on a compact connected manifold and all basic functions on $(L, \mathcal{F})$ are constant, then the evaluation maps $\mathrm{ev}_x \colon l(L, \mathcal{F}) \xrightarrow{\cong} N_x(\mathcal{F})$ are isomorphisms, and the associated canonical Maurer–Cartan form is $(\omega_{\mathrm{MC}})_x(\xi) := \mathrm{ev}_x^{-1}(\mathrm{pr}_x^N(\xi))$.

Lemma 3.47. *Let $(L_b, \mathcal{F}_b)$ and $(L_{b'}, \mathcal{F}_{b'})$ be transversely parallelizable foliations on compact connected manifolds, and assume that all basic functions on $(L_b, \mathcal{F}_b)$ and $(L_{b'}, \mathcal{F}_{b'})$ are constant.*

Let $\omega^b_{\mathrm{MC}} \in \Omega^1\big(L_b, l(L_b,\mathcal{F}_b)\big)$ *and* $\omega^{b'}_{\mathrm{MC}} \in \Omega^1\big(L_{b'}, l(L_{b'},\mathcal{F}_{b'})\big)$ *be the canonical Maurer–Cartan forms. If* $\psi : (L_b,\mathcal{F}_b) \to (L_{b'},\mathcal{F}_{b'})$ *is a foliated diffeomorphism, then*

(i) ψ_* *induces a Lie algebra isomorphism* $\psi' : l(L_b,\mathcal{F}_b) \xrightarrow{\cong} l(L_{b'},\mathcal{F}_{b'})$, $\psi'(\bar{Y}) = \overline{\psi_* Y}$ *where* $Y \in L(L_b,\mathcal{F}_b)$ *is a projectable vector field representing* $\bar{Y} \in l(L_b,\mathcal{F}_b)$.

(ii) $\psi^*(\omega^{b'}_{\mathrm{MC}}) = \psi' \circ \omega^b_{\mathrm{MC}}$.

Proof. (i) Since ψ is foliated, $d\psi$ sends $T(\mathcal{F}_b)$ into $T(\mathcal{F}_{b'})$, hence ψ_* sends $\mathfrak{X}(\mathcal{F}_b)$ into $\mathfrak{X}(\mathcal{F}_{b'})$. Let $Y \in L(L_b,\mathcal{F}_b)$ be projectable. For any $X' \in \mathfrak{X}(\mathcal{F}_{b'})$, let $X := (\psi^{-1})_* X' \in \mathfrak{X}(\mathcal{F}_b)$. Then

$$[X', \psi_* Y] = [\psi_* X, \psi_* Y] = \psi_*[X,Y] \in \psi_*\mathfrak{X}(\mathcal{F}_b) \subseteq \mathfrak{X}(\mathcal{F}_{b'}),$$

so $\psi_* Y \in L(L_{b'},\mathcal{F}_{b'})$. Therefore ψ_* induces a map $\psi' \colon l(L_b,\mathcal{F}_b) \to l(L_{b'},\mathcal{F}_{b'})$ by passing to the quotient, $\psi'(\bar{Y}) := \overline{\psi_* Y}$.

It is a Lie algebra homomorphism because ψ_* preserves Lie brackets. Its inverse is induced by $(\psi^{-1})_*$, hence ψ' is a Lie algebra isomorphism.

(ii) Let $N(\mathcal{F}_b) := T(L_b)/T(\mathcal{F}_b)$ and $N(\mathcal{F}_{b'}) := T(L_{b'})/T(\mathcal{F}_{b'})$ be the normal bundles, with projections $\mathrm{pr}^N \colon T(L_b) \to N(\mathcal{F}_b)$ and $\mathrm{pr}^{N'} \colon T(L_{b'}) \to N(\mathcal{F}_{b'})$.

Since ψ is foliated, $d\psi$ induces $(d\psi)^N_x \colon N_x(\mathcal{F}_b) \to N_{\psi(x)}(\mathcal{F}_{b'})$ by $(d\psi)^N_x([v]) := [d\psi_x(v)]$. For $x \in L_b$, let $\mathrm{ev}_x \colon l(L_b,\mathcal{F}_b) \to N_x(\mathcal{F}_b)$, $\bar{Y} \mapsto \bar{Y}_x$, be the evaluation map, and similarly $\mathrm{ev}_{\psi(x)}$ for $(L_{b'},\mathcal{F}_{b'})$. Under our hypotheses, these are linear isomorphisms, and the canonical Maurer–Cartan forms are $(\omega^b_{\mathrm{MC}})_x(\xi) = \mathrm{ev}_x^{-1}\big(\mathrm{pr}^N_x(\xi)\big)$, $(\omega^{b'}_{\mathrm{MC}})_{\psi(x)}(\zeta) = \mathrm{ev}^{-1}_{\psi(x)}\big((\mathrm{pr}^{N'})_{\psi(x)}(\zeta)\big)$.

We claim that for all $x \in L_b$,

$$\mathrm{ev}_{\psi(x)} \circ \psi' = (d\psi)^N_x \circ \mathrm{ev}_x. \tag{3.48}$$

Take $\bar{Y} \in l(L_b, \mathcal{F}_b)$ represented by a projectable Y. Then

$$\mathrm{ev}_{\psi(x)}(\psi'(\bar{Y})) = \mathrm{ev}_{\psi(x)}(\overline{\psi_* Y}) = [(\psi_* Y)_{\psi(x)}] = [d\psi_x(Y_x)] = (d\psi)^N_x([Y_x]) = (d\psi)^N_x(\mathrm{ev}_x(\bar{Y})),$$

which proves (3.48). Since ev_x and $\mathrm{ev}_{\psi(x)}$ are isomorphisms, it follows that $\mathrm{ev}^{-1}_{\psi(x)} \circ (d\psi)^N_x = \psi' \circ \mathrm{ev}^{-1}_x$.

Now let $\xi \in T_x(L_b)$. Because ψ is foliated, $(\mathrm{pr}^{N'})_{\psi(x)}(d\psi_x(\xi)) = (d\psi)^N_x(\mathrm{pr}^N_x(\xi))$. Therefore,

$$\begin{aligned}
(\psi^* \omega^{b'}_{\mathrm{MC}})_x(\xi) &= (\omega^{b'}_{\mathrm{MC}})_{\psi(x)}(d\psi_x(\xi)) \\
&= \mathrm{ev}^{-1}_{\psi(x)}\big((\mathrm{pr}^{N'})_{\psi(x)}(d\psi_x(\xi))\big) \\
&= \mathrm{ev}^{-1}_{\psi(x)}\big((d\psi)^N_x(\mathrm{pr}^N_x(\xi))\big) \\
&= \psi'\left(\mathrm{ev}^{-1}_x\big(\mathrm{pr}^N_x(\xi)\big)\right) \\
&= \psi'\big((\omega^b_{\mathrm{MC}})_x(\xi)\big),
\end{aligned}$$

which is $\psi^*(\omega^{b'}_{\mathrm{MC}}) = \psi' \circ \omega^b_{\mathrm{MC}}$. □

Proposition 3.49. *Let X be a compact manifold, let $\mathcal{F}$ be a transversely parallelizable foliation on X, let $\pi_{\mathrm{bas}} : X \to W := X/\mathcal{F}_{\mathrm{bas}}$ be the basic fibre bundle, and let $A := b(X, \mathcal{F}) \to W$ be a chosen representative of the basic Lie algebroid.*

Then there exists a section $\omega^{\pi_{\mathrm{bas}}}_{\mathrm{MC}} \in \Gamma\big((\ker d\pi_{\mathrm{bas}})^* \otimes \pi^*_{\mathrm{bas}}(\ker \rho_A)\big)$ such that, for each fibre $L_w := \pi^{-1}_{\mathrm{bas}}(w)$, the restriction $\omega^{\pi_{\mathrm{bas}}}_{\mathrm{MC}}\big|_{L_w}$ corresponds, under the restriction isomorphism $\mathrm{res}_w : (\ker \rho_A)_w \xrightarrow{\cong} l(L_w, \mathcal{F}|_{L_w})$, to the Maurer–Cartan form of the Lie foliation $(L_w, \mathcal{F}|_{L_w})$.

Proof. For $x \in X$, let $w := \pi_{\mathrm{bas}}(x)$. By the proof of Lemma 3.38, evaluation at x induces a linear isomorphism $\mathrm{ev}^X_x : A_w \longrightarrow N_x(\mathcal{F})$, $[\bar{Y}] \longmapsto \bar{Y}_x$, and, under this identification, the anchor $\rho_{A,w} : A_w \to T_w W$ corresponds to the map induced by $d\pi_{\mathrm{bas}} : N_x(\mathcal{F}) \longrightarrow T_w W$. Using local frames of A coming from a transverse parallelism, these fibrewise maps assemble to a smooth vector bundle isomorphism $\mathrm{ev}^X : \pi_{\mathrm{bas}}^* A \xrightarrow{\cong} N(\mathcal{F})$, which identifies $\pi_{\mathrm{bas}}^*(\ker \rho_A)$ with $\ker\big(d\pi_{\mathrm{bas}} : N(\mathcal{F}) \to \pi_{\mathrm{bas}}^*(TW)\big)$.

Since $T(\mathcal{F}) \subseteq \ker d\pi_{\mathrm{bas}}$, the quotient map $TX \to N(\mathcal{F}) = TX/T(\mathcal{F})$ restricts to

$$q : \ker d\pi_{\mathrm{bas}} \longrightarrow \ker\big(d\pi_{\mathrm{bas}} : N(\mathcal{F}) \to \pi_{\mathrm{bas}}^*(TW)\big).$$

Define

$$\omega_{\mathrm{MC}}^{\pi_{\mathrm{bas}}} := \big(\mathrm{ev}^X|_{\pi_{\mathrm{bas}}^*(\ker \rho_A)}\big)^{-1} \circ q.$$

So at $x \in X$, if $\xi \in \ker(d\pi_{\mathrm{bas}})_x$, then $(\omega_{\mathrm{MC}}^{\pi_{\mathrm{bas}}})_x(\xi) \in (\ker \rho_A)_w$. Because q is smooth and $\mathrm{ev}^X|_{\pi_{\mathrm{bas}}^*(\ker \rho_A)}$ is a smooth vector bundle isomorphism, the composition is smooth. So we have a smooth section of $(\ker d\pi_{\mathrm{bas}})^* \otimes \pi_{\mathrm{bas}}^*(\ker \rho_A)$.

Now let $w \in W$, write $L_w = \pi_{\mathrm{bas}}^{-1}(w)$, and note that, since π_{bas} is constant on L_w, the pullback coefficient bundle restricts to the trivial bundle

$$\pi_{\mathrm{bas}}^*(\ker \rho_A)|_{L_w} \cong L_w \times (\ker \rho_A)_w,$$

hence $\omega_{\mathrm{MC}}^{\pi_{\mathrm{bas}}}|_{L_w}$ is a $(\ker \rho_A)_w$-valued 1-form on L_w.

For $x \in L_w$, let $\mathrm{ev}_x^{L_w} : l(L_w, \mathcal{F}|_{L_w}) \longrightarrow N_x(\mathcal{F}|_{L_w})$ be the evaluation isomorphism. By Lemma 3.38, $(L_w, \mathcal{F}|_{L_w})$ is a Lie foliation and $\mathrm{res}_w : (\ker \rho_A)_w \xrightarrow{\cong} l(L_w, \mathcal{F}|_{L_w})$ is a Lie algebra isomorphism. Moreover, we have shown that $\mathrm{ev}_x^{L_w} \circ \mathrm{res}_w = \mathrm{ev}_x^X\big|_{(\ker \rho_A)_w}$.

For $\xi \in T_x L_w = \ker(d\pi_{\mathrm{bas}})_x$, let $\bar{\xi}$ denote the class of ξ in $N_x(\mathcal{F}|_{L_w}) = T_x L_w / T_x(\mathcal{F})$. Then, by construction, $\mathrm{ev}_x^X((\omega_{\mathrm{MC}}^{\pi_{\mathrm{bas}}})_x(\xi)) = q_x(\xi)$, and under the identification $\ker\big(d\pi_{\mathrm{bas}} : N_x(\mathcal{F}) \to T_w W\big) \cong N_x(\mathcal{F}|_{L_w})$ this is $\bar{\xi}$. Hence $\mathrm{ev}_x^{L_w}(\mathrm{res}_w((\omega_{\mathrm{MC}}^{\pi_{\mathrm{bas}}})_x(\xi))) = \bar{\xi}$.

Since $\mathrm{ev}_x^{L_w}$ is an isomorphism, it follows that

$$\mathrm{res}_w\big((\omega_{\mathrm{MC}}^{\pi_{\mathrm{bas}}})_x(\xi)\big) = (\mathrm{ev}_x^{L_w})^{-1}(\bar{\xi}).$$

This is the formula for the Maurer–Cartan form of the Lie foliation $(L_w, \mathcal{F}|_{L_w})$. Hence $\omega_{\mathrm{MC}}^{\pi_{\mathrm{bas}}}\big|_{L_w}$ corresponds, under res_w, to that Maurer–Cartan form. □

Remark 3.50. We want to emphasize what Proposition 3.49 says conceptually.

A priori, each fibre L_w carries its own Maurer–Cartan form, valued in the structural Lie algebra identified with $(\ker \rho_A)_w$; these are separate pieces of data on separate fibres. Proposition 3.49 says that the fibrewise Lie foliations are not isolated. Their Maurer–Cartan forms fit together into one global object $\omega_{\mathrm{MC}}^{\pi_{\mathrm{bas}}}$, and the coefficient bundle of that global object is the pullback $\pi_{\mathrm{bas}}^*(\ker \rho_A)$.

With everything we have discussed, we now give a proof of the main theorem.

Proof of Theorem 3.41. By Proposition 2.10, the given 2-metric $\eta^{(2)}$ induces a 0-metric $\eta^{(0)}$ on M. Let $g^M := \eta^{(0)}$. By Proposition 3.14, the orbitwise normal maps assemble to a smooth normal representation $\lambda^N : G \curvearrowright N$. By Proposition 3.5, this representation acts by fibrewise isometries for the induced metric g^N on N. Finally, Proposition 3.6 shows that $(M, \mathcal{O})$ is a compact connected Riemannian foliation.

(i) Definitions 3.15, 3.17, and 3.19 produce the transverse orthonormal frame bundle $\pi : OF(M, \mathcal{O}) \to M$, the lifted foliation $\widetilde{\mathcal{O}}$, the transverse canonical form θ, the transverse Levi–Civita connection ∇^{tr} on N, and the associated principal connection form ω. Proposition 3.20 shows that ω is well-defined and projectable, and Lemma 3.21 yields the vector bundle isomorphism $N(\widetilde{\mathcal{O}}) \cong OF(M, \mathcal{O}) \times \big(\mathbb{R}^q \oplus \mathfrak{o}(q)\big)$. Hence $(OF(M, \mathcal{O}), \widetilde{\mathcal{O}})$ is transversely parallelizable.

(ii) Apply Theorem 2.4 to the compact connected Riemannian foliation $(M, \mathcal{O})$. This yields a smooth manifold B with a smooth right $O(q)$-action and an $O(q)$-equivariant fibre bundle $\kappa : OF(M, \mathcal{O}) \to B$ whose fibres are the closures of the leaves of $\widetilde{\mathcal{O}}$.

(iii) Let $X := OF(M, \mathcal{O})$, $\mathcal{F} := \widetilde{\mathcal{O}}$. Since $\pi : X \to M$ is a principal $O(q)$-bundle with compact base M and compact fibre $O(q)$, the total space X is compact. By part (i), $(X, \mathcal{F})$ is transversely parallelizable. Therefore Proposition 3.37 yields a transitive basic Lie algebroid $A := b(X, \mathcal{F}) \to W := X/\mathcal{F}_{\text{bas}}$.

By [12, Corollary 4.25], the leaves of $\mathcal{F}_{\text{bas}}$ are the closures of the leaves of $\mathcal{F}$. Hence $\kappa : X \to B$ and $\pi_{\text{bas}} : X \to W$ have the same fibres. By Lemma 3.44, there is a unique diffeomorphism $\Phi : B \xrightarrow{\cong} W$ such that $\pi_{\text{bas}} = \Phi \circ \kappa$.

$$\begin{array}{ccc} & X & \\ {\scriptstyle\kappa}\swarrow & & \searrow{\scriptstyle\pi_{\text{bas}}} \\ B & \xrightarrow[\Phi]{} & W \end{array}$$

Using this diffeomorphism, we regard A as a transitive Lie algebroid over B. For each $b \in B$, let $L_b := \kappa^{-1}(b)$. By Theorem 2.4(iii), the restricted foliation $\widetilde{\mathcal{O}}|_{L_b}$ is a Lie foliation with dense holonomy group, and its structural Lie algebra is independent of b up to isomorphism. The additional identification with the isotropy Lie algebra of the basic Lie algebroid comes from Lemma 3.38, which gives a Lie algebra isomorphism $(\ker \rho_A)_b \xrightarrow{\cong} l\big(L_b, \widetilde{\mathcal{O}}|_{L_b}\big)$.

(iv) By Proposition 3.14, the normal representation $\lambda^N : G \curvearrowright N$ is smooth, and by Proposition 3.5 it acts by fibrewise isometries for g^N. Definition 3.24 and Proposition 3.25 therefore give the smooth left action of G on $OF(M, \mathcal{O})$ with moment map π, $g \cdot e = \lambda^N_g \circ e$, and show that this action commutes with the right $O(q)$-action. Its canonical character follows from Lemma 3.26.

Let $\sigma : U \to G$ be a local bisection in the sense of Definition 3.29, and set $V := t(\sigma(U))$. For the induced map $f_\sigma : \pi^{-1}(U) \to \pi^{-1}(V)$, $f_\sigma(e) := \sigma(\pi(e)) \cdot e$, Proposition 3.33 shows that f_σ preserves $\widetilde{\mathcal{O}}$, θ, and ω.

Since M and $O(q)$ are compact, the principal $O(q)$-bundle $OF(M, \mathcal{O}) \to M$ has compact total space. By (ii), B is a manifold; hence $\kappa : OF(M, \mathcal{O}) \to B$ is proper. Let $B_U := \{b \in B \mid L_b = \kappa^{-1}(b) \subseteq \pi^{-1}(U)\}$ and $B_V := \{b \in B \mid L_b = \kappa^{-1}(b) \subseteq \pi^{-1}(V)\}$. Applying Lemma 3.42 to f_σ gives a unique diffeomorphism $\bar{f}_\sigma : B_U \to B_V$ such that $\kappa \circ f_\sigma = \bar{f}_\sigma \circ \kappa$ on $\kappa^{-1}(B_U)$; equivalently, $f_\sigma(L_b) = L_{\bar{f}_\sigma(b)}$ for every $b \in B_U$.

The $O(q)$-equivariance of $\bar{f}_\sigma$ follows from Lemma 3.43(3). The compatibility with products of local bisections follows from Lemma 3.43(1)–(2), and inverse local bisections give the inverse maps. Thus the local bisections of G define the stated pseudogroup of local $O(q)$-equivariant diffeomorphisms of B. Via the diffeomorphism $\Phi : B \xrightarrow{\cong} W$ from (iii), this gives the corresponding pseudogroup of local diffeomorphisms of W.

For $b \in B_U$, let $f_{\sigma,b} := f_\sigma|_{L_b} : L_b \to L_{\bar{f}_\sigma(b)}$. The preceding paragraph shows that $f_{\sigma,b}$ is a diffeomorphism, and since f_σ preserves $\widetilde{\mathcal{O}}$, it is a foliated diffeomorphism from $(L_b, \widetilde{\mathcal{O}}|_{L_b})$ to $(L_{\bar{f}_\sigma(b)}, \widetilde{\mathcal{O}}|_{L_{\bar{f}_\sigma(b)}})$. By Lemma 3.47(i), it induces an isomorphism of structural Lie algebras $(f_{\sigma,b})_* : l(L_b, \widetilde{\mathcal{O}}|_{L_b}) \to l(L_{\bar{f}_\sigma(b)}, \widetilde{\mathcal{O}}|_{L_{\bar{f}_\sigma(b)}})$. Composing this isomorphism with the restriction isomorphisms of Lemma 3.38 gives $\alpha_{\sigma,b} := \mathrm{res}^{-1}_{\bar{f}_\sigma(b)} \circ (f_{\sigma,b})_* \circ \mathrm{res}_b : (\ker \rho_A)_b \to (\ker \rho_A)_{\bar{f}_\sigma(b)}$. $\square$

As we just discussed at the very end of the proof of Theorem 3.41, we identify the Molino base B with the basic base W and then identify the structural Lie algebra of the fibrewise foliation with the isotropy Lie algebra of the basic Lie algebroid: $(\ker \rho_A)_b \cong l\big(L_b, \widetilde{\mathcal{O}}|_{L_b}\big)$. So what A does for us is that it packages the whole Molino picture into a single object. The anchor ρ_A records infinitesimal motion along the Molino base $B \cong W$. The isotropy bundle $\ker \rho_A$ records the fibrewise structural Lie algebras. And from Proposition 3.49, the pullback of $\ker \rho_A$ to the total space is the bundle in which the fibrewise Maurer–Cartan forms live globally.

A globalizes the family of structural Lie algebras of the Molino fibres.

Without A, we only know that each fibre L_b carries a Lie foliation with its structural Lie algebra. With A, those fibrewise Lie algebras are assembled into a smooth vector bundle of Lie algebras $\ker \rho_A \subseteq A$ sitting inside one transitive Lie algebroid over the base.

3.4 The Lie algebroid of G and infinitesimal transverse isotropy

3.4.1 Construction of $A_G = \mathrm{Lie}(G)$

We introduce another Lie algebroid into the picture,

$$A_G := \mathrm{Lie}(G) \to M,$$

which is the Lie algebroid attached to the Lie groupoid $G \rightrightarrows M$. This should be distinguished from the basic Lie algebroid $A = b\big(OF(M, \mathcal{O}), \widetilde{\mathcal{O}}\big) \to B$. The beginning of our construction mirrors [12, Prop. 6.1], and later we will adapt it to our setting.

For each $x \in M$, the source fibre $s^{-1}(x) \subseteq G$ is the manifold of arrows whose source is x, and its distinguished point is the unit arrow $1_x \in G$. The fibre of the Lie algebroid at x is

$$(A_G)_x = T_{1_x}(s^{-1}(x)) = \ker(ds)_{1_x}.$$

See [2, Def. 1.19]. So an element of $(A_G)_x$ is an infinitesimal arrow leaving x. Equivalently, A_G is the pullback of the source-vertical bundle $\ker(ds) \subseteq TG$ along the unit map $u : M \to G$, $u(x) = 1_x$.

Let

$$T^sG := \ker(ds) \subseteq TG.$$

Proposition 3.51 (cf. [12, Prop. 6.1])**.** *Let $\mathfrak{X}^s_{\mathrm{inv}}(G)$ be the space of right-invariant source-vertical vector fields on G,*

$$\mathfrak{X}^s_{\mathrm{inv}}(G) := \{X \in \mathfrak{X}(G) \mid X_g \in \ker(ds)_g \quad \forall g \in G,\ X_{hg} = dR_g(X_h) \quad \forall (h,g) \text{ composable}\}.$$

Then:

(i) *$\mathfrak{X}^s_{\mathrm{inv}}(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$.*

(ii) *Restriction to the units identifies $\mathfrak{X}^s_{\mathrm{inv}}(G)$ with $\Gamma(A_G)$.*

Proof. For (i), let $X, Y \in \mathfrak{X}^s_{\mathrm{inv}}(G)$. Since X and Y are tangent to the fibres of the submersion s, their bracket is tangent to the fibres as well. Hence $[X, Y]$ is source-vertical.

It remains to check right invariance. Let $g : x \to y$. Then right multiplication by g is the diffeomorphism

$$R_g : s^{-1}(y) \to s^{-1}(x), \qquad h \mapsto hg.$$

Because X and Y are right-invariant, their restrictions to $s^{-1}(y)$ are R_g-related to their restrictions to $s^{-1}(x)$. Since R_g is a diffeomorphism, it preserves Lie brackets of related vector fields. Therefore $[X, Y]$ is R_g-related to itself, that is,

$$[X, Y]_{hg} = dR_g([X, Y]_h)$$

whenever (h, g) is composable. So $[X, Y] \in \mathfrak{X}^s_{\mathrm{inv}}(G)$.

For (ii), a right-invariant source-vertical field is determined by its values at the units, because

$$g = 1_{t(g)}g \implies X_g = dR_g(X_{1_{t(g)}}).$$

Now $X_{1_x} \in \ker(ds)_{1_x} = (A_G)_x$, so $x \mapsto X_{1_x}$ defines a smooth section of $A_G \to M$. This gives a map

$$\mathfrak{X}^s_{\mathrm{inv}}(G) \to \Gamma(A_G).$$

Conversely, if $a \in \Gamma(A_G)$, define its right-invariant extension by

$$(a^r)_g := dR_g(a_{t(g)}), \qquad g \in G.$$

This is well defined because $a_{t(g)} \in (A_G)_{t(g)} = T_{1_{t(g)}}(s^{-1}(t(g)))$, and

$$R_g : s^{-1}(t(g)) \to s^{-1}(s(g)).$$

By construction, a^r is source-vertical and right-invariant. Restriction to the units and right-invariant extension are inverse operations. Hence $\mathfrak{X}^s_{\mathrm{inv}}(G) \cong \Gamma(A_G)$. □

Remark 3.52 ([2, Def. 1.23]). Under the identification $\Gamma(A_G) \cong \mathfrak{X}^s_{\mathrm{inv}}(G)$, the bracket on $\Gamma(A_G)$ is induced from the Lie bracket of vector fields on G. The anchor is

$$\rho_{A_G} : A_G \to TM, \qquad \rho_{A_G} = dt|_{A_G}.$$

Equivalently, if $a \in \Gamma(A_G)$ and $a^r \in \mathfrak{X}^s_{\mathrm{inv}}(G)$ is its right-invariant extension, then

$$dt(a^r) = \rho_{A_G}(a) \circ t.$$

So this is the Lie algebroid attached to the Lie groupoid $G \rightrightarrows M$.

3.4.2 Bracket, anchor, and Leibniz rule for A_G

We now record explicitly the Lie algebroid structure on A_G.

Proposition 3.53 ([2, Def. 1.21, Def. 1.23, Prop. 1.24])**.** *Under the identification $\Gamma(A_G) \cong \mathfrak{X}^s_{\text{inv}}(G)$, the bracket on $\Gamma(A_G)$ is induced from the Lie bracket of right-invariant source-vertical vector fields. More precisely, if $a, b \in \Gamma(A_G)$ with right-invariant extensions a^r, b^r, then*

$$[a, b]^r = [a^r, b^r].$$

The anchor is induced by the derivative of the target map:

$$\rho_G : A_G \to TM, \qquad \rho_G(\xi) = dt(\xi).$$

Equivalently, if $a \in \Gamma(A_G)$ and a^r is its right-invariant extension, then a^r is t-projectable to $\rho_G(a)$, that is, $dt(a^r) = \rho_G(a) \circ t$.

Moreover, for $a, b \in \Gamma(A_G)$ and $f \in C^\infty(M)$, the Leibniz rule is

$$[a, fb] = f[a, b] + \rho_G(a)(f)b.$$

Proof. The bracket part is immediate from the preceding discussion: since right-invariant source-vertical vector fields form a Lie algebra and restriction to the units identifies $\mathfrak{X}^s_{\text{inv}}(G)$ with $\Gamma(A_G)$, the space $\Gamma(A_G)$ inherits a bracket. If $a, b \in \Gamma(A_G)$ with right-invariant extensions a^r, b^r, then

$$[a, b]^r = [a^r, b^r].$$

For the anchor, define $\rho_G : A_G \to TM$ by $\rho_G(\xi) = dt(\xi)$. Take $X \in \mathfrak{X}^s_{\text{inv}}(G)$, and let $g : x \to y$. Since $g = 1_y g$, we have $X_g = dR_g(X_{1_y})$. We then compute

$$\begin{aligned} dt(X_g) = dt(dR_g(X_{1_y})) &= d(t \circ R_g)(X_{1_y}) \\ &= dt(X_{1_y}), \end{aligned}$$

because for any $h \in s^{-1}(y)$, $t(R_g(h)) = t(hg) = t(h)$. We note that the right-hand side only depends on $y = t(g)$. Therefore X is t-projectable. If we define $Y \in \mathfrak{X}(M)$ by $Y_y := dt(X_{1_y})$, then $dt(X_g) = Y_{t(g)}$ for all $g \in G$.

Now let $a \in \Gamma(A_G)$ and let a^r be its right-invariant extension. By construction, $(a^r)_{1_x} = a_x$ for all $x \in M$, so the vector field Y above is exactly $\rho_G(a)$. Hence a^r is t-projectable to $\rho_G(a)$, that is,

$$dt(a^r) = \rho_G(a) \circ t.$$

Next we discuss the Leibniz rule. Let $a, b \in \Gamma(A_G)$ and $f \in C^\infty(M)$. We compute

$$\begin{aligned} [a, fb]^r &= [a^r, (fb)^r] \\ &= [a^r, (f \circ t)b^r] \quad \text{since } (fb)^r = (f \circ t)b^r \\ &= (f \circ t)[a^r, b^r] + a^r(f \circ t)b^r \\ &= (f \circ t)[a^r, b^r] + (\rho_G(a)(f) \circ t)b^r \quad \text{because } a^r \text{ projects to } \rho_G(a) \\ &= (f[a, b])^r + (\rho_G(a)(f)b)^r. \end{aligned}$$

Since the right-invariant extension map is injective, we conclude

$$[a, fb] = f[a, b] + \rho_G(a)(f)b.$$

□

Remark 3.54. Fibrewise, $\rho_{G,x} : (A_G)_x = \ker(ds)_{1_x} \to T_xM$ is the map $\rho_{G,x} = dt_{1_x}|_{\ker(ds)_{1_x}}$. So an element of $(A_G)_x$ is an infinitesimal arrow starting at x, and the anchor tells us how the target endpoint moves to first order.

Also, since a^r and b^r are t-related to $\rho_G(a)$ and $\rho_G(b)$, the anchor is automatically a Lie algebra homomorphism:

$$\rho_G([a,b]) = [\rho_G(a), \rho_G(b)].$$

At this point we have the full Lie algebroid structure attached to the groupoid: $(A_G)_x = \ker(ds)_{1_x}$, $\rho_G = dt|_{A_G}$, with bracket induced from right-invariant source-vertical vector fields.

3.4.3 How A_G recovers the orbit foliation

We now explain how $A_G = \mathrm{Lie}(G)$ recovers the orbit foliation $\mathcal{O}$. After we discuss the basic constructions, this is the starting place where A_G begins to explain Molino theory.

Proposition 3.55. *Let $G \rightrightarrows M$ be a regular Lie groupoid, let $A_G = \mathrm{Lie}(G) \to M$ be its Lie algebroid, and let $\rho_G : A_G \to TM$ be the anchor. For each $x \in M$, let O_x be the orbit through x and let $G_x := s^{-1}(x) \cap t^{-1}(x)$ be the isotropy group at x. Then*

$$\mathrm{Im}(\rho_{G,x}) = T_x(O_x) \qquad \textit{and} \qquad (\ker \rho_G)_x = \mathrm{Lie}(G_x).$$

Consequently,

$$\mathrm{Im}(\rho_G) = T(\mathcal{O}),$$

and there is a short exact sequence of vector bundles

$$0 \longrightarrow \ker \rho_G \longrightarrow A_G \xrightarrow{\rho_G} T(\mathcal{O}) \longrightarrow 0.$$

Proof. We first show that the anchor recovers the orbit directions. Let $x \in M$. By Lemma 3.7, the restriction

$$t_x := t|_{s^{-1}(x)} : s^{-1}(x) \to O_x$$

is a surjective submersion. Differentiating at the unit 1_x, and using $(A_G)_x = T_{1_x}(s^{-1}(x)) = \ker(ds)_{1_x}$, we get

$$d(t_x)_{1_x} : (A_G)_x \to T_x(O_x).$$

But $d(t_x)_{1_x} = dt_{1_x}|_{\ker(ds)_{1_x}} = \rho_{G,x}$ since t_x is the restriction of t. Hence $\mathrm{Im}(\rho_{G,x}) = T_x(O_x)$.

So fibrewise we have the picture that $\mathrm{Im}(\rho_{G,x}) = T_x(O_x)$. Since G is regular, the orbit foliation $\mathcal{O}$ has constant rank, so these spaces fit together into the tangent bundle $T(\mathcal{O}) \subseteq TM$. Thus

$$\mathrm{Im}(\rho_G) = T(\mathcal{O}).$$

Next we compute the isotropy of A_G. The isotropy group at x is $G_x = s^{-1}(x) \cap t^{-1}(x)$. Its tangent space at the unit is

$$T_{1_x} G_x = T_{1_x}(s^{-1}(x) \cap t^{-1}(x)) = \ker(ds)_{1_x} \cap \ker(dt)_{1_x}.$$

But $(A_G)_x = \ker(ds)_{1_x}$ and $\rho_{G,x} = dt_{1_x}|_{\ker(ds)_{1_x}}$. Hence

$$(\ker \rho_G)_x = \ker(ds)_{1_x} \cap \ker(dt)_{1_x} = T_{1_x} G_x = \mathrm{Lie}(G_x).$$

Since $\mathrm{Im}(\rho_G) = T(\mathcal{O})$ is a vector subbundle of TM, the anchor has constant rank. Therefore we obtain the exact sequence

$$0 \longrightarrow \ker \rho_G \longrightarrow A_G \xrightarrow{\rho_G} T(\mathcal{O}) \longrightarrow 0.$$

□

Remark 3.56. So A_G infinitesimally generates the orbit foliation. At the same time, its isotropy is exactly the bundle of isotropy Lie algebras. In this sense, A_G is the foliation together with the bundle of isotropy Lie algebras.

This is the first place where $\mathrm{Lie}(G)$ starts to explain Molino theory.

3.4.4 The isotropy of A_G on transverse directions

Recall the discussion from Lemma 3.7 through Proposition 3.14. We observe that the normal representation is built from ds and dt, which are exactly the maps that define $A_G = \mathrm{Lie}(G)$. Molino theory is about transverse geometry, and the transverse geometry of the orbit foliation is already encoded infinitesimally in the source/target differentials. The Lie algebroid A_G controls the infinitesimal motion tangent to the orbits, while the normal representation λ^N controls how arrows act on directions transverse to the orbits.

Proposition 3.57. *For each $x \in M$, let $G_x = s^{-1}(x) \cap t^{-1}(x)$ be the isotropy group at x. Then the isotropy representation*

$$\lambda^N_x : G_x \longrightarrow O(N_x, g^N_x), \qquad g \longmapsto \lambda^N_g,$$

differentiates to a Lie algebra representation

$$d\lambda^N_x : (\ker \rho_G)_x = \mathrm{Lie}(G_x) \longrightarrow \mathfrak{so}(N_x, g^N_x).$$

In particular, the isotropy part of A_G acts infinitesimally on the transverse directions.

Proof. If $g \in G_x$, then $\lambda_g^N : N_x \to N_x$. So each isotropy group G_x acts linearly on the normal space N_x. By Proposition 3.5, this action is orthogonal for g_x^N, so we get a Lie group homomorphism

$$\lambda_x^N : G_x \longrightarrow O(N_x, g_x^N).$$

Take $\xi \in \mathrm{Lie}(G_x) = T_{1_x} G_x = (\ker \rho_G)_x$. Choose a smooth curve $g(t) \in G_x$ with $g(0) = 1_x$ and $g'(0) = \xi$. Define

$$d\lambda_x^N(\xi)(v) := \left.\frac{d}{dt}\right|_{t=0} \lambda_{g(t)}^N(v), \qquad v \in N_x.$$

Since differentiation of a Lie group homomorphism is a Lie algebra homomorphism, $d\lambda_x^N$ is a Lie algebra representation on N_x.

Since every $\lambda_{g(t)}^N$ is orthogonal, for any $v, w \in N_x$ we have

$$g_x^N(\lambda_{g(t)}^N v, \lambda_{g(t)}^N w) = g_x^N(v, w).$$

Differentiate at $t = 0$ to get

$$0 = \left.\frac{d}{dt}\right|_{t=0} g_x^N(\lambda_{g(t)}^N v, \lambda_{g(t)}^N w) = g_x^N(d\lambda_x^N(\xi)v, w) + g_x^N(v, d\lambda_x^N(\xi)w).$$

So $d\lambda_x^N(\xi)$ is skew-adjoint relative to g_x^N. Hence $d\lambda_x^N(\xi) \in \mathfrak{so}(N_x, g_x^N)$. □

Remark 3.58. What we can say here is that the isotropy part of A_G does not just sit abstractly in the kernel of the anchor; it acts infinitesimally on the transverse directions.

Remark 3.59. Compare this with the basic Lie algebroid $A \to W$ coming from the lifted foliation, or equivalently with the corresponding Lie algebroid over B after identifying $B \cong W$.

There we had

$$0 \longrightarrow \ker \rho_A \longrightarrow A \xrightarrow{\rho_A} TW \longrightarrow 0,$$

and here we have

$$0 \longrightarrow \ker \rho_G \longrightarrow A_G \xrightarrow{\rho_G} T(\mathcal{O}) \longrightarrow 0.$$

So ρ_G gives the leaf/orbit directions on M, while ρ_A gives directions on the closure base W. The isotropy $(\ker \rho_G)_x = \mathrm{Lie}(G_x)$ is the pointwise isotropy Lie algebra of the original groupoid, while $(\ker \rho_A)_w$, equivalently $(\ker \rho_A)_b$ after identifying $B \cong W$, is the structural Lie algebra of the Lie foliation on the closure fibre L_b.

This is an important distinction, and let me put it more directly. $A_G = \mathrm{Lie}(G) \to M$ is the infinitesimal symmetry before taking closures, while $A = b\big(OF(M, \mathcal{O}), \widetilde{\mathcal{O}}\big) \to B$ is the infinitesimal symmetry after passing to the Molino closure picture.

So $A_G \to M$ explains why Molino's theory starts the way it does, and $A \to B$ is the Lie algebroid that Molino's theory produces.

Remark 3.60. This also fits the usual Lie group/Lie algebra picture. For a Lie group H, the Lie algebra is $\mathrm{Lie}(H) = T_eH$, and left- or right-invariant vector fields identify global infinitesimal symmetries with tangent vectors at the identity.

For a Lie groupoid $G \rightrightarrows M$, there is no single identity, but there is one unit 1_x for each $x \in M$. So the infinitesimal object is no longer one vector space; it is a vector bundle

$$x \longmapsto T_{1_x}(s^{-1}(x)).$$

This explains why Lie algebroids are vector bundles over the same base as the groupoid.

The bracket plays the role of the differentiated multiplication law: it is obtained from the Lie bracket of right-invariant source-vertical vector fields. The target map $t : G \to M$ tells us where an arrow ends, so differentiating t along the source-vertical directions at the units gives the anchor

$$\rho_G = dt|_{\ker(ds)}.$$

The Leibniz rule is the compatibility condition we computed earlier in this section.

If $\Phi : H \to G$ is a homomorphism of Lie groupoids, then differentiating at the units gives a morphism of Lie algebroids

$$\mathrm{Lie}(\Phi) : \mathrm{Lie}(H) \to \mathrm{Lie}(G).$$

So there is a differentiation process

$$\text{Lie groupoids} \longrightarrow \text{Lie algebroids},$$

which is to say that a Lie algebroid is the infinitesimal part of a Lie groupoid [2, Ex. 28].

Chapter 4

Corollaries of Molino's theory

4.1 Compact-group equivariant case

We discuss a G_{cpt}-equivariant Molino structure theorem. The reason we elaborate on compact-group equivariance is that it provides a toy model for the lifting and descent constructions used later in the proofs of our corollaries.

We first average the transverse metric and then analyze the lifted action on the transverse orthonormal frame bundle.

Let $(M, \mathcal{F})$ be a smooth foliated manifold of codimension q. From [12, Sec. 2.2], a transverse metric on $(M, \mathcal{F})$ is a positive $C^\infty(M)$-bilinear form g on $\mathfrak{X}(M)$ such that $\ker(g_x) = T_x\mathcal{F}$ for any $x \in M$, and $L_X g = 0$ for any vector field X on M tangent to $\mathcal{F}$.

Remark 4.1. By [12, Remark 2.4], the condition $\ker(g_x) = T_x\mathcal{F}$ means that g is the pull-back of a Riemannian structure on the normal bundle $N(\mathcal{F}) = TM/T(\mathcal{F})$ along the quotient map

$TM \to N(\mathcal{F})$. We will use this identification and denote again by g the induced fibre metric on $N(\mathcal{F})$.

Let G_{cpt} be a compact Lie group acting smoothly on $(M, \mathcal{F})$ by foliated diffeomorphisms. Thus, for each $a \in G_{\text{cpt}}$,

$$(da)_x\big(T_x\mathcal{F}\big) = T_{a\cdot x}\mathcal{F}, \qquad x \in M,$$

and hence $(da)_x$ induces linear isomorphisms on the normal fibres

$$(da)_x^N : N_x(\mathcal{F}) \to N_{a\cdot x}(\mathcal{F}).$$

Lemma 4.2. *Let μ be the normalized Haar measure on G_{cpt}. Define $g_{\text{av}} := \int_{G_{\text{cpt}}} a^*g \, d\mu(a)$. That is, for $X, Y \in \mathfrak{X}(M)$ and $x \in M$,*

$$g_{\text{av}}(X,Y)(x) = \int_{G_{\text{cpt}}} (a^*g)(X,Y)(x) \, d\mu(a).$$

Then g_{av} is a transverse metric on $(M, \mathcal{F})$ and it is G_{cpt}-invariant. Under the identification of Remark 4.1, the induced fibre metric on $N(\mathcal{F})$ is given pointwise by

$$(g_{\text{av}})_x(\nu, \nu') = \int_{G_{\text{cpt}}} g_{a\cdot x}\Big((da)_x^N(\nu), (da)_x^N(\nu')\Big) \, d\mu(a), \qquad \nu, \nu' \in N_x(\mathcal{F}).$$

Moreover, there exists a smooth G_{cpt}-invariant Riemannian metric ρ on M which is bundle-like with respect to $\mathcal{F}$ and whose associated transverse metric in the sense of [12, Remark 2.7(7)] *equals g_{av}.*

Proof. Step 1: Set up.

For $X, Y \in \mathfrak{X}(M)$, the map $(a, x) \longmapsto (a^*g)(X, Y)(x)$ is smooth on $G_{\text{cpt}} \times M$. Since G_{cpt} is compact, the Haar integral defines a smooth function $g_{\text{av}}(X, Y) \in C^\infty(M)$. Clearly, symmetry, bilinearity, and nonnegativity are preserved by integration.

Let $x \in M$. If $\xi \in T_x\mathcal{F}$, then $(da)_x\xi \in T_{a\cdot x}\mathcal{F} = \ker(g_{a\cdot x})$ for all $a \in G_{\text{cpt}}$, hence $(a^*g)_x(\xi, -) = 0$ for all a, and therefore $(g_{\text{av}})_x(\xi, -) = 0$. Thus $T_x\mathcal{F} \subseteq \ker((g_{\text{av}})_x)$. Conversely, if $\xi \in \ker((g_{\text{av}})_x)$, then $(g_{\text{av}})_x(\xi, \xi) = 0$, so

$$0 = \int_{G_{\text{cpt}}} g_{a\cdot x}\big((da)_x\xi, (da)_x\xi\big)\, d\mu(a).$$

Step 2: g_{av} is a transverse metric.

The integrand is a continuous nonnegative function of $a \in G_{\text{cpt}}$, hence must vanish identically. Taking $a = 1_{G_{\text{cpt}}}$ gives $g_x(\xi, \xi) = 0$, so $\xi \in \ker(g_x) = T_x\mathcal{F}$. Therefore $\ker((g_{\text{av}})_x) = T_x\mathcal{F}$.

For the Lie derivative condition, let $X \in \mathfrak{X}(\mathcal{F})$. Using the naturality of Lie derivatives under pullback, $L_X(a^*g) = a^*(L_{a_*X}g)$. Since the action is foliated, $a_*X \in \mathfrak{X}(\mathcal{F})$, and since g is transverse, $L_{a_*X}g = 0$. Hence $L_X(a^*g) = 0$ for all $a \in G_{\text{cpt}}$, and therefore

$$L_X g_{\text{av}} = \int_{G_{\text{cpt}}} L_X(a^*g)\, d\mu(a) = 0.$$

Thus g_{av} is a transverse metric on $(M, \mathcal{F})$.

Step 3: Averaged metric on the normal bundle.

Under the identification of Remark 4.1, if $\nu = [v]$ and $\nu' = [v']$ in $N_x(\mathcal{F})$, we compute

$$(a^*g)_x(v, v') = g_{a\cdot x}\big((da)_x^N(\nu), (da)_x^N(\nu')\big).$$

Integrating over G_{cpt} gives the desired formula for $(g_{\mathrm{av}})_x(\nu, \nu')$.

We check that g_{av} is G_{cpt}-invariant. To do so, let $b \in G_{\mathrm{cpt}}$. Using right-invariance of the Haar measure, we compute

$$b^* g_{\mathrm{av}} = \int_{G_{\mathrm{cpt}}} b^*(a^* g)\, d\mu(a) = \int_{G_{\mathrm{cpt}}} (ab)^* g\, d\mu(a) = \int_{G_{\mathrm{cpt}}} a'^* g\, d\mu(a') = g_{\mathrm{av}}.$$

Step 4: Construct an invariant Riemannian metric.

For the last part of the lemma, we choose an arbitrary Riemannian metric η_0 on M and average it over G_{cpt}, as in the proof of [12, Prop. 2.8], to obtain a G_{cpt}-invariant Riemannian metric η. Let $H := T(\mathcal{F})^{\perp_\eta} \subseteq TM$. Since $a_*(T(\mathcal{F})) = T(\mathcal{F})$ and $a^*\eta = \eta$ for every $a \in G_{\mathrm{cpt}}$, the subbundle H is G_{cpt}-invariant. Let $\mathrm{pr} : TM \to N(\mathcal{F})$ be the quotient map. The restriction

$$\mathrm{pr}|_H : H \xrightarrow{\cong} N(\mathcal{F})$$

is a vector bundle isomorphism. Define a Riemannian metric ρ on $TM = T(\mathcal{F}) \oplus H$ by declaring this decomposition ρ-orthogonal and setting

$$\rho|_{T(\mathcal{F})} := \eta|_{T(\mathcal{F})}, \qquad \rho|_H := (\mathrm{pr}|_H)^* g_{\mathrm{av}}.$$

The metric ρ is smooth. It is G_{cpt}-invariant because $\eta|_{T(\mathcal{F})}$ is G_{cpt}-invariant, H is G_{cpt}-invariant, and $\mathrm{pr}|_H$ is G_{cpt}-equivariant while g_{av} is G_{cpt}-invariant on $N(\mathcal{F})$. Finally, the associated transverse metric of ρ is g_{av}: if $X, Y \in \mathfrak{X}(M)$ and $X^{(n)}, Y^{(n)}$ denote their ρ-normal components, then

$$g_\rho(X, Y) = \rho\big(X^{(n)}, Y^{(n)}\big) = g_{\mathrm{av}}(X, Y),$$

because $T(\mathcal{F})^{\perp_\rho} = H$ and $\rho|_H$ is the pullback of g_{av} along $\mathrm{pr}|_H$. Since g_{av} is a transverse metric, ρ is bundle-like with respect to $\mathcal{F}$. □

Replacing g by g_{av}, we will assume that $a^*g = g$ for all $a \in G_{\mathrm{cpt}}$. For all $x \in M$, the induced map

$$(da)^N_x : \big(N_x(\mathcal{F}), g_x\big) \to \big(N_{a\cdot x}(\mathcal{F}), g_{a\cdot x}\big)$$

is an isometry.

We recall that, following [12, Example 4.19], $F(M,\mathcal{F})$ denotes the transverse frame bundle of $N(\mathcal{F})$. A point of $F_x(M,\mathcal{F})$ is a linear isomorphism $e : \mathbb{R}^q \xrightarrow{\cong} N_x(\mathcal{F})$, and the right $GL(q,\mathbb{R})$-action is given by composition $eA = e \circ A$. The transverse orthonormal frame bundle $OF(M,\mathcal{F}) \subseteq F(M,\mathcal{F})$ is the subbundle consisting of the orthogonal isomorphisms.

Lemma 4.3. *Let $\pi : F(M,\mathcal{F}) \to M$ be the projection. For $a \in G_{\mathrm{cpt}}$ define*

$$\tilde{a} : F(M,\mathcal{F}) \to F(M,\mathcal{F}), \qquad \tilde{a}(e) := (da)^N_{\pi(e)} \circ e,$$

where $e : \mathbb{R}^q \to N_{\pi(e)}(\mathcal{F})$. Then $a \mapsto \tilde{a}$ defines a smooth left G_{cpt}-action on $F(M,\mathcal{F})$ by principal $GL(q,\mathbb{R})$-bundle isomorphisms.

If $a^*g = g$ for all $a \in G_{\mathrm{cpt}}$, then each $(da)^N_x$ is an isometry of $\big(N_x(\mathcal{F}), g_x\big)$, hence $\tilde{a}$ preserves $OF(M,\mathcal{F})$ and restricts to

$$\widehat{a} : OF(M,\mathcal{F}) \to OF(M,\mathcal{F}), \qquad \widehat{a}(e) := (da)^N_{\pi(e)} \circ e.$$

The resulting G_{cpt}-action on $OF(M,\mathcal{F})$ commutes with the right $O(q)$-action.

Proof. Step 1: $(da)^N_x$ exists.

Let $a, b \in G_{\mathrm{cpt}}$ and $e \in F(M, \mathcal{F})$, and write $x := \pi(e)$. Since the action is foliated, each $(da)_x$ maps $T_x\mathcal{F}$ to $T_{a\cdot x}\mathcal{F}$, so it induces a linear isomorphism $(da)_x^N : N_x(\mathcal{F}) \to N_{a\cdot x}(\mathcal{F})$.

Step 2: Smoothness.

Because $G_{\mathrm{cpt}} \times M \to M$ is smooth, the induced map $(a, x) \longmapsto (da)_x^N$ is smooth, and hence so is $(a, e) \longmapsto \tilde{a}(e) = (da)_{\pi(e)}^N \circ e$.

Step 3: Left action and principal $GL(q, \mathbb{R})$-equivariance.

Moreover,

$$\widetilde{ab}(e) = (d(ab))_x^N \circ e = (da)_{b\cdot x}^N \circ (db)_x^N \circ e = \tilde{a}(\tilde{b}(e)), \qquad \widetilde{1_{G_{\mathrm{cpt}}}}(e) = e,$$

so $a \mapsto \tilde{a}$ is a left G_{cpt}-action on $F(M, \mathcal{F})$. Also,

$$\pi(\tilde{a}(e)) = a \cdot \pi(e).$$

By [12, Example 4.19], the right $GL(q, \mathbb{R})$-action is given by composition $eA = e \circ A$. Hence for any $A \in GL(q, \mathbb{R})$,

$$\tilde{a}(eA) = (da)_{\pi(e)}^N \circ e \circ A = \tilde{a}(e)A.$$

Thus each $\tilde{a}$ is a principal $GL(q, \mathbb{R})$-bundle isomorphism, with inverse $\widetilde{a^{-1}}$.

Step 4: The action preserves orthonormal frames.

Assume $a^*g = g$. Then for all $x \in M$, $(da)_x^N$ is a fibrewise isometry of $(N(\mathcal{F}), g)$. Therefore, if $e : \mathbb{R}^q \to N_x(\mathcal{F})$ is an orthogonal isomorphism, then

$$(da)_x^N \circ e : \mathbb{R}^q \to N_{a\cdot x}(\mathcal{F})$$

is again an orthogonal isomorphism, so $\tilde{a}$ preserves $OF(M,\mathcal{F})$ and defines $\widehat{a}$.

Finally, for $A \in O(q)$,

$$\widehat{a}(eA) = (da)^N_{\pi(e)} \circ e \circ A = \widehat{a}(e)A,$$

so the left G_{cpt}-action commutes with the right $O(q)$-action. □

For the next two lemmas, it is convenient to work first on the transverse frame bundle $F(M,\mathcal{F})$. Recall the notation from Theorem 3.41. Let $\theta^F \in \Omega^1(F(M,\mathcal{F}),\mathbb{R}^q)$ be the transverse canonical form. This is defined by $\theta^F_e(\xi) := e^{-1}\big(\overline{(d\pi)_e(\xi)}\big)$ for all $e : \mathbb{R}^q \to N_{\pi(e)}(\mathcal{F})$ and $\xi \in T_eF(M,\mathcal{F})$.

Its restriction to $OF(M,\mathcal{F})$ is the transverse canonical form $\theta := \theta^F\big|_{OF(M,\mathcal{F})}$ used in Proposition 4.7.

Lemma 4.4. *For each $a \in G_{\mathrm{cpt}}$, the lift $\tilde{a} : F(M,\mathcal{F}) \to F(M,\mathcal{F})$ satisfies $\tilde{a}^*\theta^F = \theta^F$. In particular, $\widehat{a}^*\theta = \theta$ on $OF(M,\mathcal{F})$.*

Proof. Let $a \in G_{\mathrm{cpt}}$, $e \in F(M,\mathcal{F})$, and $\xi \in T_eF(M,\mathcal{F})$, and write $x := \pi(e)$. For $v \in T_xM$, write $\overline{v} \in N_x(\mathcal{F}) = T_xM/T_x\mathcal{F}$ for its class in the normal fibre.

Since $\pi \circ \tilde{a} = a \circ \pi$, differentiating at e gives $(d\pi)_{\tilde{a}(e)}\big((d\tilde{a})_e(\xi)\big) = (da)_x\big((d\pi)_e(\xi)\big) \in T_{a\cdot x}M$. Projecting to the normal bundle and using that a preserves $\mathcal{F}$, we obtain

$$\overline{(d\pi)_{\tilde{a}(e)}\big((d\tilde{a})_e(\xi)\big)} = (da)^N_x\big(\overline{(d\pi)_e(\xi)}\big) \in N_{a\cdot x}(\mathcal{F}).$$

Therefore,

$$\begin{aligned}
(\tilde{a}^*\theta^F)_e(\xi) &= \theta^F_{\tilde{a}(e)}\big((d\tilde{a})_e(\xi)\big) \\
&= \big((da)^N_x \circ e\big)^{-1}\left(\overline{(d\pi)_{\tilde{a}(e)}\big((d\tilde{a})_e(\xi)\big)}\right) \\
&= \big((da)^N_x \circ e\big)^{-1}\left((da)^N_x\big(\overline{(d\pi)_e(\xi)}\big)\right) \\
&= e^{-1}\big(\overline{(d\pi)_e(\xi)}\big) \\
&= \theta^F_e(\xi).
\end{aligned}$$

Hence $\tilde{a}^*\theta^F = \theta^F$. Restricting to $OF(M, \mathcal{F})$ gives $\widehat{a}^*\theta = \theta$. □

Let ∇^{tr} be the metric connection on $N(\mathcal{F})$ determined by g, and let $\omega \in \Omega^1(OF(M, \mathcal{F}), \mathfrak{o}(q))$ be the associated principal $O(q)$-connection form on $OF(M, \mathcal{F})$.

Recall that we had a similar discussion earlier in Proposition 3.20. We choose a Haefliger cocycle $(s_i : U_i \to \mathbb{R}^q)$ so that $g|_{U_i} = s_i^* g_i$ for a Riemannian metric g_i on $s_i(U_i)$, and all transition diffeomorphisms are isometries. Let ω_i be the ordinary Levi–Civita connection form on the frame bundle $F(s_i(U_i))$, and define

$$\widetilde{s}_i^F : F(M, \mathcal{F})|_{U_i} \to F(s_i(U_i)), \qquad \widetilde{s}_i^F(e) := (ds_i)^N_{\pi(e)} \circ e.$$

These local forms glue to a projectable $GL(q, \mathbb{R})$-connection $\omega^F \in \Omega^1(F(M, \mathcal{F}), \mathfrak{gl}(q, \mathbb{R}))$ characterized by

$$\omega^F\big|_{F(M,\mathcal{F})|_{U_i}} = (\widetilde{s}_i^F)^*\omega_i.$$

Its restriction to $OF(M, \mathcal{F})$ is exactly the principal $O(q)$-connection form ω associated with ∇^{tr}.

Lemma 4.5. *For each $a \in G_{\mathrm{cpt}}$, the lift $\tilde{a} : F(M,\mathcal{F}) \to F(M,\mathcal{F})$ satisfies $\tilde{a}^*\omega^F = \omega^F$. In particular, $\widehat{a}^*\omega = \omega$ on $OF(M,\mathcal{F})$.*

Proof. Fix $a \in G_{\mathrm{cpt}}$. Let i be an index and let $e \in \tilde{a}^{-1}\big(F(M,\mathcal{F})|_{U_i}\big)$. Write $x := \pi(e)$. Choose j with $x \in U_j$, and shrink if necessary to an open neighbourhood $U \subseteq U_j \cap a^{-1}(U_i)$ of x such that the restricted submersion $s_j|_U : U \to s_j(U)$ has connected fibres.

Since a is foliated, after shrinking U, $a|_U$ sends each connected plaque of the chart $s_j|_U$ into a connected plaque of the chart $s_i|_{a(U)}$, so $s_i \circ a$ is constant on the fibres of $s_j|_U$. Hence there exists a unique smooth map $\varphi_{a,ij} : s_j(U) \to s_i(a(U))$ such that $s_i \circ a = \varphi_{a,ij} \circ s_j$ on U.

Because $a^*g = g$, $g|_U = s_j^* g_j$, and $g|_{a(U)} = s_i^* g_i$, we compute

$$s_j^* g_j = g|_U = a^*(g|_{a(U)}) = a^*(s_i^* g_i) = (\varphi_{a,ij} \circ s_j)^* g_i = s_j^*(\varphi_{a,ij}^* g_i).$$

Since $s_j|_U$ admits local smooth sections, it follows that $\varphi_{a,ij}^* g_i = g_j$ on $s_j(U)$. Thus $\varphi_{a,ij}$ is an isometry, hence induces a smooth map $F(\varphi_{a,ij}) : F(s_j(U)) \to F(s_i(a(U)))$. By invariance of the Levi–Civita connection under isometries [7, Prop. 5.13], $F(\varphi_{a,ij})^*\big(\omega_i|_{F(s_i(a(U)))}\big) = \omega_j|_{F(s_j(U))}$.

Now let $\widetilde{U} := F(M,\mathcal{F})|_U \subseteq F(M,\mathcal{F})|_{U_j}$. Differentiating $s_i \circ a = \varphi_{a,ij} \circ s_j$ and passing to normal bundles gives $(ds_i)^N_{a\cdot x} \circ (da)^N_x = (d\varphi_{a,ij})_{s_j(x)} \circ (ds_j)^N_x$. Therefore $\widetilde{s}_i^F \circ \tilde{a} = F(\varphi_{a,ij}) \circ \widetilde{s}_j^F$

on $\widetilde{U}$. Using $\omega^F\big|_{F(M,\mathcal{F})|_{U_i}} = (\widetilde{s}_i^F)^*\omega_i$ and $\omega^F\big|_{F(M,\mathcal{F})|_{U_j}} = (\widetilde{s}_j^F)^*\omega_j$, we obtain on $\widetilde{U}$:

$$\begin{aligned}(\tilde{a}^*\omega^F)|_{\widetilde{U}} = \tilde{a}^*\big(\omega^F|_{F(M,\mathcal{F})|_{U_i}}\big) &= \tilde{a}^*\big((\widetilde{s}_i^F)^*\omega_i\big)\\ &= (\widetilde{s}_i^F \circ \tilde{a})^*\omega_i\\ &= (F(\varphi_{a,ij}) \circ \widetilde{s}_j^F)^*\omega_i\\ &= (\widetilde{s}_j^F)^*\big(F(\varphi_{a,ij})^*\omega_i\big)\\ &= (\widetilde{s}_j^F)^*\omega_j = \omega^F|_{\widetilde{U}}.\end{aligned}$$

Since these open sets $\widetilde{U}$ cover $F(M,\mathcal{F})$, we conclude that $\tilde{a}^*\omega^F = \omega^F$. Restricting to $OF(M,\mathcal{F})$ gives $\widehat{a}^*\omega = \omega$. □

Corollary 4.6. *The lifted G_{cpt}-actions on $F(M,\mathcal{F})$ and $OF(M,\mathcal{F})$ preserve the lifted foliation $\widetilde{\mathcal{F}}$.*

Proof. By [12, Example 4.19], the transverse canonical form on the transverse frame bundle satisfies $\ker((\theta^F)_e) = \ker(d\pi)_e \oplus T_e(\widetilde{\mathcal{F}})$ for $e \in F(M,\mathcal{F})$. Also, by construction, ω^F is a projectable connection on the transverse principal $GL(q,\mathbb{R})$-bundle $(F(M,\mathcal{F}),\widetilde{\mathcal{F}})$, hence $T_e(\widetilde{\mathcal{F}}) \subseteq \ker((\omega^F)_e)$.

Let $V_e := \ker(d\pi)_e$ be the vertical space, and let $\mathcal{H}_e := \ker((\omega^F)_e)$ be the horizontal space. Since ω^F is a principal connection form on $F(M,\mathcal{F})$, we have $T_e(F(M,\mathcal{F})) = V_e \oplus \mathcal{H}_e$ and $V_e \cap \mathcal{H}_e = \{0\}$.

Now take $\xi \in \ker((\theta^F)_e) \cap \ker((\omega^F)_e)$. Using $\ker((\theta^F)_e) = V_e \oplus T_e(\widetilde{\mathcal{F}})$, write $\xi = v + \xi_{\widetilde{\mathcal{F}}}$, where $v \in V_e$ and $\xi_{\widetilde{\mathcal{F}}} \in T_e(\widetilde{\mathcal{F}})$. Because ω^F vanishes on $T(\widetilde{\mathcal{F}})$, $0 = (\omega^F)_e(\xi) = (\omega^F)_e(v) + (\omega^F)_e(\xi_{\widetilde{\mathcal{F}}}) = (\omega^F)_e(v)$, so $v \in V_e \cap \mathcal{H}_e = \{0\}$. Hence $\xi = \xi_{\widetilde{\mathcal{F}}} \in T_e(\widetilde{\mathcal{F}})$, and therefore

$$T_e(\widetilde{\mathcal{F}}) = \ker((\theta^F)_e) \cap \ker((\omega^F)_e).$$

Finally, by Lemmas 4.4 and 4.5, $\tilde{a}^*\theta^F = \theta^F$ and $\tilde{a}^*\omega^F = \omega^F$. Hence $(d\tilde{a})_e\big(\ker((\theta^F)_e)\big) = \ker((\theta^F)_{\tilde{a}(e)})$, $(d\tilde{a})_e\big(\ker((\omega^F)_e)\big) = \ker((\omega^F)_{\tilde{a}(e)})$, and so

$$(d\tilde{a})_e\big(T_e(\widetilde{\mathcal{F}})\big) = (d\tilde{a})_e\Big(\ker((\theta^F)_e) \cap \ker((\omega^F)_e)\Big) = \ker((\theta^F)_{\tilde{a}(e)}) \cap \ker((\omega^F)_{\tilde{a}(e)}) = T_{\tilde{a}(e)}(\widetilde{\mathcal{F}}).$$

Thus the lifted G_{cpt}-action on $F(M, \mathcal{F})$ preserves $\widetilde{\mathcal{F}}$. Restricting to the invariant subbundle $OF(M, \mathcal{F})$ gives the same conclusion there. □

Proposition 4.7 (G_{cpt}-equivariant)**.** *Let $(M, \mathcal{F})$ be a smooth foliated manifold of codimension q, and let $N(\mathcal{F}) = TM/T(\mathcal{F})$ denote its normal bundle. Let g be a transverse metric on $(M, \mathcal{F})$, so that $(\mathcal{F}, g)$ is a Riemannian foliation on M.*

Assume that a compact Lie group G_{cpt} acts smoothly on $(M, \mathcal{F})$ by foliated diffeomorphisms. Then, after replacing g by its G_{cpt}-average and choosing a G_{cpt}-invariant bundle-like Riemannian metric on M whose associated transverse metric is this average, we may assume that the G_{cpt}-action is by foliated isometries of $(\mathcal{F}, g)$, i.e. $a^*g = g$ for all $a \in G_{\mathrm{cpt}}$.

The induced G_{cpt}-action on the transverse orthonormal frame bundle $\pi : OF(M, \mathcal{F}) \to M$ commutes with the right $O(q)$-action and preserves the lifted foliation $\widetilde{\mathcal{F}}$, the transverse canonical form θ, and the transverse Levi–Civita connection form ω. In particular, when M is compact and connected and we apply Molino's structure theorem to $(\mathcal{F}, g)$, the resulting $O(q)$-equivariant fibre bundle $\kappa : OF(M, \mathcal{F}) \to B$ is G_{cpt}-equivariant for a unique induced smooth G_{cpt}-action on B commuting with the $O(q)$-action.

Proof. It remains to discuss the induced action on B. Assume now that M is compact and connected. By Theorem 2.4(ii), there exists a manifold B with a right $O(q)$-action and an $O(q)$-equivariant fibre bundle $\kappa : OF(M, \mathcal{F}) \to B$ whose fibres are exactly the closures of the leaves of $\widetilde{\mathcal{F}}$.

Since each $\widehat{a}$ preserves $\widetilde{\mathcal{F}}$, it sends leaves to leaves and hence leaf closures to leaf closures. Therefore $\kappa \circ \widehat{a}$ is constant on the fibres of κ. Because κ is a fibre bundle, for each $a \in G_{\mathrm{cpt}}$ there exists a unique smooth map $\bar{a} : B \to B$ such that $\kappa \circ \widehat{a} = \bar{a} \circ \kappa$. Applying the same construction to a^{-1} shows that $\overline{a^{-1}}$ is the inverse of $\bar{a}$, so each $\bar{a}$ is a diffeomorphism.

Uniqueness immediately gives $\overline{ab} = \bar{a} \circ \bar{b}$ and $\overline{1_{G_{\mathrm{cpt}}}} = \mathrm{id}_B$, so $a \mapsto \bar{a}$ defines an action of G_{cpt} on B. To prove smoothness of the action map, consider

$$\Phi : G_{\mathrm{cpt}} \times OF(M, \mathcal{F}) \to B, \qquad \Phi(a, e) = \kappa(\widehat{a}(e)).$$

This map is smooth because $(a, e) \mapsto \widehat{a}(e)$ and κ are smooth. Let $b_0 \in B$. Since κ is a fibre bundle, there exist an open neighbourhood $U \ni b_0$ and a smooth local section $\sigma : U \to OF(M, \mathcal{F})$. Then on $G_{\mathrm{cpt}} \times U$,

$$(a, b) \longmapsto \bar{a}(b) = \Phi(a, \sigma(b)) = \kappa\big(\widehat{a}(\sigma(b))\big)$$

is smooth. Hence the action map $G_{\mathrm{cpt}} \times B \to B$ is smooth.

Finally, since $\widehat{a}$ commutes with the right $O(q)$-action and κ is $O(q)$-equivariant, the induced G_{cpt}-action on B commutes with the $O(q)$-action. For all $A \in O(q)$ and $e \in OF(M, \mathcal{F})$,

$$\bar{a}\big(\kappa(e)A\big) = \bar{a}\big(\kappa(eA)\big) = \kappa\big(\widehat{a}(eA)\big) = \kappa\big(\widehat{a}(e)A\big) = \kappa\big(\widehat{a}(e)\big)A = \bar{a}\big(\kappa(e)\big)A.$$

By uniqueness, this is the unique smooth G_{cpt}-action on B for which κ is G_{cpt}-equivariant. $\square$

4.2 The orbifold case

With the machinery developed in the compact-group-equivariant case, we now construct a Molino-type theory for orbifolds.

Remark 4.8. The author found related results in Lin–Miyamoto's work on Riemannian foliations and quasifolds [9]. Their work studies leaf spaces of Killing Riemannian foliations from the diffeological point of view; in particular, under certain completeness hypotheses, the leaf space of a Killing Riemannian foliation is a diffeological quasifold.

Our perspective here is complementary. We start with an orbifold equipped with a Riemannian foliation and construct the Molino data intrinsically on its transverse orthonormal frame bundle.

Definition 4.9. Let Q be an orbifold and let $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i\in I}$ be an orbifold atlas. A foliation $\mathcal{F}_Q$ of codimension q on Q is given by foliations $\mathcal{F}_i$ of codimension q on the manifolds U_i such that every embedding of orbifold charts $\lambda : (U_i, G_i, \phi_i) \to (U_j, G_j, \phi_j)$ is foliated.

After shrinking the orbifold charts if necessary, we assume that each U_i is a simple foliation chart for $\mathcal{F}_i$, so that $\mathcal{F}_i$ is defined on U_i by a submersion $s_i : U_i \to T_i \subseteq \mathbb{R}^q$ with connected fibres. Then every embedding of orbifold charts λ satisfies $s_j \circ \lambda = \gamma_\lambda \circ s_i$ for some local diffeomorphism γ_λ.

A transverse metric on $(Q, \mathcal{F}_Q)$ is a collection $\bar{g} = (\bar{g}_i)$ of transverse metrics on the local foliations $(U_i, \mathcal{F}_i)$ such that every embedding of orbifold charts is a transverse isometry. A Riemannian foliation on Q is a foliation $\mathcal{F}_Q$ together with a transverse metric $\bar{g}$.

We briefly explain the construction. Let $(\mathcal{F}_Q, \bar{g})$ be a Riemannian foliation on the orbifold Q. For each chart (U_i, G_i, ϕ_i), let $\pi_i : OF(U_i, \mathcal{F}_i) \to U_i$ be the transverse orthonormal frame bundle of the manifold foliation $(U_i, \mathcal{F}_i)$, equipped with its lifted foliation $\widetilde{\mathcal{F}}_i$, transverse canonical form θ_i, and transverse Levi–Civita connection form ω_i.

By Lemma 3.23, every embedding of orbifold charts induces an $O(q)$-equivariant foliated embedding between the corresponding transverse orthonormal frame bundles, preserving the local forms θ_i and ω_i. Hence the quotient orbifolds $OF(U_i, \mathcal{F}_i)/G_i$, together with the induced maps, glue, as in the construction of $OF(Q)$, to an orbifold, which we denote by $OF(Q, \mathcal{F}_Q)$.

It carries a natural right $O(q)$-action. The descended local lifted foliations glue to an orbifold foliation on $OF(Q, \mathcal{F}_Q)$, denoted $\widetilde{\mathcal{F}}_Q$, and the descended local forms glue to orbifold 1-forms on $OF(Q, \mathcal{F}_Q)$.

With the orbifold frame bundle notation fixed, the orbifold analogue of Molino's theorem reads as follows.

Corollary 4.10 (Orbifold Molino's structure theorem)**.** *Let Q be a compact connected Riemannian orbifold, and let $(\mathcal{F}_Q, \bar{g})$ be a Riemannian foliation of codimension q on Q. Let $\mathcal{O} := OF(Q, \mathcal{F}_Q)$ with its natural right $O(q)$-action, and let $\widetilde{\mathcal{F}}$ be the lifted foliation on $\mathcal{O}$.*

Then there exist orbifold 1-forms θ, ω on $\mathcal{O}$, a manifold $\mathcal{N}$ with a smooth right $O(q)$-action, an $O(q)$-equivariant orbifold fibre bundle $\bar{s} : \mathcal{O} \to \mathcal{N}$, and a Lie algebra $\mathfrak{g}$ such that:

(1) $T(\widetilde{\mathcal{F}}) = \ker\theta \cap \ker\omega$. In particular, $(\mathcal{O}, \widetilde{\mathcal{F}})$ is transversely parallelizable.

(2) The fibres of $\bar{s}$ are exactly the closures of the leaves of $\widetilde{\mathcal{F}}$.

(3) For every $z \in \mathcal{N}$, the restricted foliation $\widetilde{\mathcal{F}}\big|_{\bar{s}^{-1}(z)}$ is a Lie foliation defined by a $\mathfrak{g}$-valued Maurer–Cartan form with dense holonomy group, and $\mathfrak{g}$ is independent of z up to isomorphism.

Manifold presentation and pullback foliation

Lemma 4.11. *Let Q be an orbifold of dimension n equipped with a Riemannian metric. Then its unitary frame bundle $M := UF(Q)$ is a smooth manifold with a smooth action of the compact connected Lie group $U(n)$ which is transitive on the fibres of $p : M \to Q$ and has finite isotropy groups, such that $Q \cong M/U(n)$ as orbifolds. If Q is compact (resp. connected), then M is compact (resp. connected).*

Proof. Following [12, Sec. 2.4], let $\rho = (\rho_i)$ denote the given Riemannian metric on the orbifold Q, and let $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i\in I}$ be a maximal orbifold atlas of Q, so that each $U_i \subseteq \mathbb{R}^n$ is a connected open set, each G_i is a finite subgroup of $\mathrm{Diff}(U_i)$, and any embedding of orbifold charts $\lambda : (U_i, G_i, \phi_i) \to (U_j, G_j, \phi_j)$ is an isometric embedding $(U_i, \rho_i) \to (U_j, \rho_j)$.

Complexify the tangent spaces:

For each i, set $T(U_i)^{\mathbb{C}} := TU_i \otimes \mathbb{C}$. We extend the Riemannian metric ρ_i to a Hermitian inner product on the complexified tangent bundle by $\langle v \otimes z, w \otimes z' \rangle := z\overline{z'}\rho_i(v, w)$, where $x \in U_i$, $v, w \in T_xU_i$, and $z, z' \in \mathbb{C}$.

Then define

$$UF(U_i) := \left\{ e : \mathbb{C}^n \to T_x(U_i)^{\mathbb{C}} \;\middle|\; x \in U_i,\ e \text{ is unitary} \right\},$$

with $\pi_i : UF(U_i) \to U_i$ given by $\pi_i(e) = x$ when $e : \mathbb{C}^n \to T_x(U_i)^{\mathbb{C}}$. This is a principal $U(n)$-bundle over U_i.

Lift the group action to the frame bundle:

There is a smooth left G_i-action on $UF(U_i)$ given by

$$g \cdot e := \big((dg)_x \otimes \mathrm{id}_{\mathbb{C}}\big) \circ e \in UF_{g(x)}(U_i), \qquad g \in G_i,\ e \in UF_x(U_i),$$

which commutes with the right $U(n)$-action on frames.

We claim that the G_i-action on $UF(U_i)$ is free. To see this, if $g \cdot e = e$, then $g(x) = x$ and $(dg)_x^{\mathbb{C}} = \mathrm{id}$, hence $(dg)_x = \mathrm{id}$. Since U_i is connected and G_i is finite, Lemma 2.10 of [12] implies that $g = \mathrm{id}$. Therefore $UF(U_i)/G_i$ is a smooth manifold.

Lift embeddings of orbifold charts:

If $\lambda : (U_i, G_i, \phi_i) \to (U_j, G_j, \phi_j)$ is an embedding of orbifold charts, then $d\lambda \otimes \mathrm{id}_{\mathbb{C}}$ induces a smooth $U(n)$-equivariant embedding $\widetilde{\lambda} : UF(U_i) \to UF(U_j)$. Let $\overline{\lambda} : G_i \to G_j$ be the associated homomorphism from [12, Prop. 2.12(i)]. Then for $g \in G_i$ and $e \in UF(U_i)$, we compute

$$\widetilde{\lambda}(g \cdot e) = \big((d\lambda)_{g(x)} \otimes \mathrm{id}_{\mathbb{C}}\big) \circ \big((dg)_x \otimes \mathrm{id}_{\mathbb{C}}\big) \circ e = \big(d(\overline{\lambda}(g))_{\lambda(x)} \otimes \mathrm{id}_{\mathbb{C}}\big) \circ \widetilde{\lambda}(e) = \overline{\lambda}(g) \cdot \widetilde{\lambda}(e),$$

so $\widetilde{\lambda}$ descends to a smooth open embedding of quotients

$$\lambda_* : UF(U_i)/G_i \longrightarrow UF(U_j)/G_j.$$

The descended map does not depend on the chosen embedding:

Note that for any $h \in G_j$, the induced map $h_* : UF(U_j)/G_j \to UF(U_j)/G_j$ is the identity. It follows that for any two embeddings $\lambda, \mu : (U_i, G_i, \phi_i) \to (U_j, G_j, \phi_j)$ we have $\lambda_* = \mu_*$ [12, Prop. 2.12(iv)].

Gluing:

Thus $\{UF(U_i)/G_i, \lambda_*\}$ forms a filtered direct system, and we define

$$M := UF(Q) := \lim_{\rightarrow}\{UF(U_i)/G_i, \lambda_*\}.$$

Then M is a smooth manifold, and each $UF(U_i)/G_i$ embeds as an open submanifold of M. The maps ϕ_i induce maps $p_i : UF(U_i)/G_i \to Q$ by $p_i([e]) = \phi_i(\pi_i(e))$, and these glue to an open map $p : M \longrightarrow Q$. The $U(n)$-action on the local pieces glues to a smooth $U(n)$-action on M along the fibres of p. By construction, Q is isomorphic to the orbifold associated to this $U(n)$-action; equivalently, by [12, Prop. 2.23], $Q \cong M/U(n)$ as orbifolds.

Fibres, transitivity, finite isotropy:

Let $q \in Q$ and $m \in p^{-1}(q)$. The $U(n)$-action is transitive on $p^{-1}(q)$, so the orbit map induces a diffeomorphism

$$p^{-1}(q) \cong U(n)/U(n)_m,$$

where $U(n)_m$ is the isotropy subgroup at m. These isotropy subgroups are finite. Since $U(n)$ is connected, it follows that each fibre $p^{-1}(q)$ is connected.

Compactness, connectedness:

Each fibre $p^{-1}(q) \cong U(n)/U(n)_m$ is compact because $U(n)$ is compact. Moreover, the action of a compact group is proper, hence the quotient map $p : M \to M/U(n) \cong Q$ is closed. Since M is a manifold (hence locally compact Hausdorff), a continuous closed map with compact fibres is proper; hence p is proper. If Q is compact, then M is compact as the preimage of the compact set Q under the proper map p.

As a quotient map for a group action, p is open. For the sake of contradiction, if M were disconnected, then $M = A \sqcup B$ with A, B nonempty disjoint open subsets. By connectedness of the fibres, for each $q \in Q$ the fibre $p^{-1}(q)$ is contained entirely in either A or B. Hence $p(A)$ and $p(B)$ are disjoint. Since p is open and surjective, $p(A)$ and $p(B)$ are nonempty open subsets of Q, and $Q = p(A) \cup p(B)$, contradicting the connectedness of Q. Thus if Q is connected, then M is connected. □

Remark 4.12. For the discussion below, $T(\mathcal{F}_Q)$, $N(\mathcal{F}_Q)$, pullbacks of these orbifold bundles, and pullback foliations along smooth maps from manifolds to Q are understood chartwise from the local foliations $(U_i, \mathcal{F}_i)$ and the local submersions s_i.

Lemma 4.13. *Let $(\mathcal{F}_Q, \bar{g})$ be a Riemannian foliation of codimension q on Q, and let $p : M := UF(Q) \to Q$ be the unitary frame bundle projection. With the setup from Definition 4.9, for each i, let $\pi_i : UF(U_i) \to U_i$ and $q_i : UF(U_i) \to W_i := UF(U_i)/G_i \subseteq M$ be the frame-bundle projection and the quotient map, respectively, and define $\tilde{s}_i := s_i \circ \pi_i : UF(U_i) \to T_i$. Then the foliation on $UF(U_i)$ defined by $\tilde{s}_i$ is preserved by the G_i-action and descends through q_i to a foliation on W_i. These descended foliations agree on overlaps and therefore glue to a foliation $\mathcal{F}_M$ of codimension q on M.*

Moreover,

(a) *dp maps $T(\mathcal{F}_M)$ into $T(\mathcal{F}_Q)$ and, chartwise, $v \in T_x\mathcal{F}_M \iff dp_x(v) \in T_{p(x)}\mathcal{F}_Q$. Equivalently, if $x = q_i(\tilde{x})$ and $v = (dq_i)_{\tilde{x}}(\tilde{v})$, then $v \in T_x\mathcal{F}_M \iff (d\pi_i)_{\tilde{x}}(\tilde{v}) \in T_{\pi_i(\tilde{x})}\mathcal{F}_i$;*

(b) *dp induces a vector bundle isomorphism $\overline{dp} : N(\mathcal{F}_M) \xrightarrow{\cong} p^*N(\mathcal{F}_Q)$;*

(c) *pulling back the transverse metric gives a transverse metric $g_M^T := \overline{dp}^{\,*}\bar{g}$ on $\mathcal{F}_M$, so $(\mathcal{F}_M, g_M^T)$ is a Riemannian foliation;*

(d) *the $U(n)$-action on M preserves $\mathcal{F}_M$ and g_M^T, and $\overline{dp}$ is $U(n)$-equivariant.*

Proof. For each i, let $\mathcal{F}_i^M$ be the foliation on $UF(U_i)$ defined by the submersion $\tilde{s}_i = s_i \circ \pi_i$. If $g \in G_i$, let $\widetilde{g} : UF(U_i) \to UF(U_i)$ be the induced action. Since g is an embedding of the orbifold chart into itself, there is a local diffeomorphism γ_g such that $s_i \circ g = \gamma_g \circ s_i$.

Because $\pi_i \circ \widetilde{g} = g \circ \pi_i$, we get

$$\tilde{s}_i \circ \widetilde{g} = s_i \circ \pi_i \circ \widetilde{g} = s_i \circ g \circ \pi_i = \gamma_g \circ s_i \circ \pi_i = \gamma_g \circ \tilde{s}_i.$$

Hence $\widetilde{g}$ preserves $\mathcal{F}_i^M$. Since the G_i-action on $UF(U_i)$ is free and finite, $\mathcal{F}_i^M$ descends through q_i to a foliation on $W_i = UF(U_i)/G_i$.

If $\lambda : (U_i, G_i, \phi_i) \to (U_j, G_j, \phi_j)$ is an embedding of orbifold charts, let $\widetilde{\lambda} : UF(U_i) \to UF(U_j)$ be the induced map. Since $\pi_j \circ \widetilde{\lambda} = \lambda \circ \pi_i$, we obtain

$$\tilde{s}_j \circ \widetilde{\lambda} = s_j \circ \pi_j \circ \widetilde{\lambda} = s_j \circ \lambda \circ \pi_i = \gamma_\lambda \circ s_i \circ \pi_i = \gamma_\lambda \circ \tilde{s}_i.$$

Thus the descended foliations on the W_i agree on overlaps and glue to a global foliation $\mathcal{F}_M$ on M.

(a) Let $x = q_i(\tilde{x}) \in W_i$ and $v = (dq_i)_{\tilde{x}}(\tilde{v}) \in T_xM$. Since q_i is a local diffeomorphism and $\mathcal{F}_M|_{W_i}$ is the quotient of $\mathcal{F}_i^M$, we have

$$v \in T_x\mathcal{F}_M \iff \tilde{v} \in T_{\tilde{x}}\mathcal{F}_i^M \iff \tilde{v} \in \ker(d\tilde{s}_i)_{\tilde{x}}.$$

Because $\tilde{s}_i = s_i \circ \pi_i$,

$$\ker(d\tilde{s}_i)_{\tilde{x}} = (d\pi_i)_{\tilde{x}}^{-1}\big(\ker(ds_i)_{\pi_i(\tilde{x})}\big) = (d\pi_i)_{\tilde{x}}^{-1}\big(T_{\pi_i(\tilde{x})}\mathcal{F}_i\big).$$

This gives the chartwise characterization of $T(\mathcal{F}_M)$ and shows that dp maps $T(\mathcal{F}_M)$ into $T(\mathcal{F}_Q)$ and that $T_x\mathcal{F}_M = (dp_x)^{-1}\big(T_{p(x)}\mathcal{F}_Q\big)$.

(b) On each local model $UF(U_i)$, the foliation $\mathcal{F}_i^M$ is the pullback of $\mathcal{F}_i$ along the submersion π_i, so $d\pi_i$ induces a vector bundle isomorphism $\overline{d\pi_i} : N(\mathcal{F}_i^M) \xrightarrow{\cong} \pi_i^* N(\mathcal{F}_i)$.

These local isomorphisms are G_i-equivariant and compatible with chart embeddings, hence descend through q_i and glue to a global vector bundle isomorphism

$$\overline{dp} : N(\mathcal{F}_M) \xrightarrow{\cong} p^* N(\mathcal{F}_Q).$$

(c) We pull back the local transverse metrics on $(U_i, \mathcal{F}_i)$ along $\overline{d\pi_i}$ to obtain transverse metrics on $(UF(U_i), \mathcal{F}_i^M)$. These are G_i-invariant, compatible under chart embeddings, and therefore descend and glue to the transverse metric

$$g_M^T := \overline{dp}^{\,*}\bar{g}$$

on $\mathcal{F}_M$. Thus $(\mathcal{F}_M, g_M^T)$ is a Riemannian foliation.

(d) The right $U(n)$-action on each $UF(U_i)$ is fibrewise with respect to π_i, so $\tilde{s}_i = s_i \circ \pi_i$ is $U(n)$-invariant. Hence the $U(n)$-action preserves each local foliation $\mathcal{F}_i^M$, and therefore preserves the glued foliation $\mathcal{F}_M$ on M.

The local bundle isomorphisms $\overline{d\pi_i}$ are $U(n)$-equivariant, so the glued map $\overline{dp}$ is $U(n)$-equivariant. Since $g_M^T = \overline{dp}^{\,*}\bar{g}$ by (c), the $U(n)$-action preserves g_M^T as well. □

Proposition 4.14. *Let $p : M = UF(Q) \to Q$ and $(\mathcal{F}_M, g_M^T)$ be as in Lemma 4.13. Let $\pi : F(M, \mathcal{F}_M) \to M$ be the transverse frame bundle, and let $\widetilde{\mathcal{F}}_M$ denote its lifted foliation; by restriction, we use the same notation for the lifted foliation on $OF(M, \mathcal{F}_M)$.*

Let H be a connected Lie group acting smoothly on M. For $a \in H$, write $\varphi_a(x) = a \cdot x$. Assume that the action is fibrewise with respect to p, so that $p \circ \varphi_a = p$ for all $a \in H$. Then,

(a) for every $x \in M$, the orbit $H \cdot x$ is contained in the leaf of $\mathcal{F}_M$ through x;

(b) the induced H-action on the transverse frame bundle $\pi : F(M, \mathcal{F}_M) \to M$ has the property that for every $e \in F(M, \mathcal{F}_M)$, the orbit $H \cdot e$ is contained in the leaf of the lifted foliation $\widetilde{\mathcal{F}}_M$ through e;

(c) if the H-action preserves the transverse metric g_M^T, so that the transverse orthonormal frame bundle $OF(M, \mathcal{F}_M) \subseteq F(M, \mathcal{F}_M)$ is H-invariant, then for every $e \in OF(M, \mathcal{F}_M)$, the orbit $H \cdot e$ is contained in the leaf of the restricted lifted foliation $\widetilde{\mathcal{F}}_M$ through e.

Proof. We first note that each φ_a is a foliated diffeomorphism of $(M, \mathcal{F}_M)$. If $v \in T_x\mathcal{F}_M$, then by Lemma 4.13(a) we have $dp_x(v) \in T_{p(x)}\mathcal{F}_Q$. Since $p \circ \varphi_a = p$,

$$dp_{a \cdot x}\big(d\varphi_a(v)\big) = d(p \circ \varphi_a)_x(v) = dp_x(v) \in T_{p(x)}\mathcal{F}_Q = T_{p(a \cdot x)}\mathcal{F}_Q.$$

Applying Lemma 4.13(a) again gives $d\varphi_a(v) \in T_{a \cdot x}\mathcal{F}_M$. Thus φ_a preserves $\mathcal{F}_M$, so it induces

$$(d\varphi_a)^N : N_x(\mathcal{F}_M) \longrightarrow N_{a \cdot x}(\mathcal{F}_M)$$

and hence an action on $F(M, \mathcal{F}_M)$. Also, if the action preserves g_M^T, then each $(d\varphi_a)^N$ is an isometry, so $OF(M, \mathcal{F}_M) \subseteq F(M, \mathcal{F}_M)$ is H-invariant.

(a) Let $x \in M$. Since $p \circ \varphi_a = p$ for all $a \in H$, we have $H \cdot x \subseteq p^{-1}(p(x))$. By Lemma 4.13(a),

$$T_y\mathcal{F}_M = (dp_y)^{-1}\big(T_{p(y)}\mathcal{F}_Q\big) \implies \ker(dp_y) \subseteq T_y\mathcal{F}_M \qquad \text{for all } y \in M.$$

This tells us that each connected component of a fibre $p^{-1}(q)$ is contained in a leaf of $\mathcal{F}_M$. Since H is connected, $H \cdot x$ is connected, and hence it is contained in the leaf of $\mathcal{F}_M$ through x.

(b) Take a Haefliger cocycle $s_i : U_i \to T_i$ for $\mathcal{F}_Q$. Then, in the chartwise sense, the pullback foliation is locally defined by $\tilde{s}_i := s_i \circ p : p^{-1}(U_i) \to T_i$. So inside the chart $p^{-1}(U_i)$, the leaves of $\mathcal{F}_M$ are the connected components of the fibres of $\tilde{s}_i$.

By (a), each φ_a preserves the leaves of $\mathcal{F}_M$. Hence the H-action induces an action on $F(M, \mathcal{F}_M)$.

Let $a \in H$ and $e \in F(M, \mathcal{F}_M)$, and write $x := \pi(e)$. Choose i such that $p(x)$ is represented in the chart U_i. Since $p(a \cdot x) = p(x)$, both x and $a \cdot x$ lie in the same local chart $p^{-1}(U_i)$. Inside that chart, the lifted foliation is locally defined by

$$\tilde{s}_i^F : F(M, \mathcal{F}_M)|_{p^{-1}(U_i)} \to F(T_i), \quad \tilde{s}_i^F(e') = (d\tilde{s}_i)^N_{\pi(e')} \circ e'.$$

Then,

$$\begin{aligned} \tilde{s}_i^F(a \cdot e) &= (d\tilde{s}_i)^N_{\pi(a \cdot e)} \circ (a \cdot e) \\ &= (d\tilde{s}_i)^N_{a \cdot x} \circ (d\varphi_a)^N_x \circ e \\ &= (d(\tilde{s}_i \circ \varphi_a))^N_x \circ e \\ &= (d\tilde{s}_i)^N_x \circ e = \tilde{s}_i^F(e). \end{aligned}$$

It follows that e and $a \cdot e$ lie in the same fibre of the local submersion $\tilde{s}_i^F$. To see that they are in the same connected component of that fibre, we can choose a path $a_t \in H$, $t \in [0, 1]$, with $a_0 = 1$ and $a_1 = a$, such that $t \mapsto a_t \cdot e$ is continuous in $F(M, \mathcal{F}_M)$. For every t, $p(\pi(a_t \cdot e)) = p(a_t \cdot x) = p(x)$, so the whole path stays in the same local chart

$F(M, \mathcal{F}_M)|_{p^{-1}(U_i)}$. It's easy to check that

$$\tilde{s}_i^F(a_t \cdot e) = \tilde{s}_i^F(e) \text{ for all } t.$$

Since this is true for all $a \in H$, $H \cdot e$ is contained in the leaf of $\widetilde{\mathcal{F}}_M$ through e.

(c) If the H-action preserves g_M^T, then each

$$(d\varphi_a)_x^N : N_x(\mathcal{F}_M) \to N_{a\cdot x}(\mathcal{F}_M)$$

is an isometry. Hence $OF(M, \mathcal{F}_M) \subseteq F(M, \mathcal{F}_M)$ is H-invariant. Again by [12, Ex. 4.19], the lifted foliation on $OF(M, \mathcal{F}_M)$ is the restriction of the lifted foliation on $F(M, \mathcal{F}_M)$. Restricting the argument in (b), we are done. □

Descent to the orbifold frame bundle

Lemma 4.15. *Let a compact Lie group G_{cpt} act smoothly on a manifold M, and let*

$$\Pi : M \to M/G_{\text{cpt}}$$

be the quotient projection. Then for every subset $A \subseteq M$, $\Pi(\overline{A}) = \overline{\Pi(A)}$, where $\overline{A}$ is the closure of A in M and $\overline{\Pi(A)}$ is the closure of $\Pi(A)$ in M/G_{cpt}.

Proof. Since Π is continuous, $\Pi(\overline{A}) \subseteq \overline{\Pi(A)}$.

For the reverse inclusion, it suffices to show that Π is a closed map, because then $\Pi(\overline{A})$ is closed in M/G_{cpt} and contains $\Pi(A)$, hence contains $\overline{\Pi(A)}$.

Let $C \subseteq M$ be closed. Since Π is a quotient map, $B \subseteq M/G_{\text{cpt}}$ is closed if and only if $\Pi^{-1}(B)$ is closed in M. Therefore $\Pi(C)$ is closed in M/G_{cpt} if and only if $\Pi^{-1}\big(\Pi(C)\big)$ is closed in M. But $\Pi^{-1}\big(\Pi(C)\big)$ is the G_{cpt}-saturation of C: $\Pi^{-1}\big(\Pi(C)\big) = G_{\text{cpt}} \cdot C$.

Let

$$\mu : G_{\text{cpt}} \times M \to M, \qquad \mu(k, x) = k \cdot x$$

be the action map. Then $G_{\text{cpt}} \cdot C = \mu(G_{\text{cpt}} \times C)$.

We claim that μ is proper. Let $L \subseteq M$ be compact. Then $\mu^{-1}(L) \subseteq G_{\text{cpt}} \times (G_{\text{cpt}} \cdot L)$. Since G_{cpt} is compact, $G_{\text{cpt}} \times L$ is compact, and hence

$$G_{\text{cpt}} \cdot L = \mu(G_{\text{cpt}} \times L)$$

is compact. As M is Hausdorff, L is closed, and therefore $\mu^{-1}(L)$ is closed in $G_{\text{cpt}} \times M$, hence closed in the compact set $G_{\text{cpt}} \times (G_{\text{cpt}} \cdot L)$. Thus $\mu^{-1}(L)$ is compact, so μ is proper. In particular, any proper map between Hausdorff manifolds is closed, hence μ is closed. Therefore

$$G_{\text{cpt}} \cdot C = \mu(G_{\text{cpt}} \times C)$$

is closed in M. This shows that $\Pi^{-1}\big(\Pi(C)\big)$ is closed, hence $\Pi(C)$ is closed in M/G_{cpt}.

Now let $C = \overline{A}$, so $\Pi(\overline{A})$ is closed and contains $\Pi(A)$, hence contains $\overline{\Pi(A)}$. Combined with the first inclusion, we are done. □

Lemma 4.16. *Let $p : M = UF(Q) \to Q$ and $(\mathcal{F}_M, g_M^T)$ be as in Lemma 4.13. Define*

$$\Xi : OF(M, \mathcal{F}_M) \longrightarrow OF(Q, \mathcal{F}_Q), \qquad \Xi(e) := \overline{dp}_{\pi(e)} \circ e,$$

where $\pi : OF(M, \mathcal{F}_M) \to M$ and $\pi_{\mathcal{O}} : OF(Q, \mathcal{F}_Q) \to Q$. Then:

(a) Ξ *is well-defined;*

(b) Ξ *is* $U(n)$*-invariant and* $O(q)$*-equivariant;*

(c) Ξ *induces an isomorphism of orbifold principal* $O(q)$*-bundles*

$$\overline{\Xi} : OF(M, \mathcal{F}_M)/U(n) \xrightarrow{\cong} OF(Q, \mathcal{F}_Q);$$

(d) under this identification, the foliation descended from the lifted foliation on $OF(M, \mathcal{F}_M)$ *is identified with the lifted foliation on* $OF(Q, \mathcal{F}_Q)$.

Proof. Let $e \in OF(M, \mathcal{F}_M)$ and $x := \pi(e)$. By Lemma 4.13(b), $\overline{dp}_x : N_x(\mathcal{F}_M) \to N_{p(x)}(\mathcal{F}_Q)$ is an isomorphism, and since $g_M^T = \overline{dp}^* \bar{g}$ it is an isometry:

$$\bar{g}_{p(x)}\big(\overline{dp}_x(\nu), \overline{dp}_x(\nu')\big) = (g_M^T)_x(\nu, \nu'), \qquad \nu, \nu' \in N_x(\mathcal{F}_M).$$

(a) Since $e : \mathbb{R}^q \to N_x(\mathcal{F}_M)$ is a g_M^T-orthogonal isomorphism, to show that $\overline{dp}_x \circ e : \mathbb{R}^q \to N_{p(x)}(\mathcal{F}_Q)$ is $\bar{g}$-orthogonal, that is, $\Xi(e) \in OF(Q, \mathcal{F}_Q)$, it remains to check orthogonality.

Take $v, w \in \mathbb{R}^q$. Then

$$\begin{aligned} \bar{g}_{p(x)}((\overline{dp}_x \circ e)(v), (\overline{dp}_x \circ e)(w)) &= \bar{g}_{p(x)}(\overline{dp}_x(e(v)), \overline{dp}_x(e(w))) \\ &= (g_M^T)_x(e(v), e(w)), \end{aligned}$$

because $\overline{dp}_x$ is an isometry. Since e is orthonormal, $\overline{dp}_x \circ e$ is also orthonormal.

(b) Let $u \in U(n)$ and let $\varphi_u : M \to M$ be given by $x \mapsto u \cdot x$. The lifted action on $OF(M, \mathcal{F}_M)$ is given by $u \cdot e = (d\varphi_u)^N_{\pi(e)} \circ e$, and $\pi(u \cdot e) = u \cdot \pi(e)$. For $e \in OF(M, \mathcal{F}_M)$ with $x := \pi(e)$, Lemma 4.13(d) gives $\overline{dp}_{u\cdot x} \circ (d\varphi_u)_x^N = \overline{dp}_x$. Therefore

$$\Xi(u \cdot e) = \overline{dp}_{u\cdot x} \circ (d\varphi_u)_x^N \circ e = \overline{dp}_x \circ e = \Xi(e),$$

so Ξ is $U(n)$-invariant.

For $A \in O(q)$,

$$\Xi(e \cdot A) = \overline{dp}_x \circ (e \circ A) = (\overline{dp}_x \circ e) \circ A = \Xi(e) \cdot A,$$

so Ξ is $O(q)$-equivariant.

(c) *Step 1*: $OF(M, \mathcal{F}_M)/U(n)$ is an orbifold.

The $U(n)$-action on $OF(M, \mathcal{F}_M)$ is proper (since $U(n)$ is compact) and locally free. If $u \cdot e = e$, then $u \cdot \pi(e) = \pi(e)$, so u lies in the finite isotropy group of $\pi(e)$ for the $U(n)$-action on M. Hence the orbit space $OF(M, \mathcal{F}_M)/U(n)$ carries the canonical orbifold structure associated to a proper locally free action, and the quotient map $\Pi : OF(M, \mathcal{F}_M) \to OF(M, \mathcal{F}_M)/U(n)$ is an orbifold submersion.

Step 2: The right $O(q)$-action descends

Since the right $O(q)$-action on $OF(M, \mathcal{F}_M)$ commutes with the left $U(n)$-action, it descends to a right $O(q)$-action on $OF(M, \mathcal{F}_M)/U(n)$.

Step 3: Ξ factors through the quotient

Because Ξ is smooth and $U(n)$-invariant, there is a unique orbifold map

$$\overline{\Xi} : OF(M, \mathcal{F}_M)/U(n) \longrightarrow OF(Q, \mathcal{F}_Q)$$

such that $\Xi = \overline{\Xi} \circ \Pi$.

Define $\Lambda : OF(Q, \mathcal{F}_Q) \to OF(M, \mathcal{F}_M)/U(n)$ as follows: for $\bar{e} \in OF(Q, \mathcal{F}_Q)$ with $q := \pi_{\mathcal{O}}(\bar{e})$, choose any $x \in p^{-1}(q)$ and let

$$e_x := \overline{dp}_x^{-1} \circ \bar{e} \in OF(M, \mathcal{F}_M), \qquad \Lambda(\bar{e}) := [e_x].$$

This is well-defined: if $x' = u \cdot x$ for some $u \in U(n)$, then by Lemma 4.13(d), $\overline{dp}_{x'}^{-1} = (d\varphi_u)_x^N \circ \overline{dp}_x^{-1}$, hence $e_{x'} = u \cdot e_x$ and therefore $[e_{x'}] = [e_x]$. Smoothness of Λ follows from the description on orbifold charts, using local trivializations for the $U(n)$-action.

Step 4: $\overline{\Xi}$ and Λ are inverse maps

By construction,

$$(\overline{\Xi} \circ \Lambda)(\bar{e}) = \overline{\Xi}([e_x]) = \Xi(e_x) = \overline{dp}_x \circ \overline{dp}_x^{-1} \circ \bar{e} = \bar{e},$$

so $\overline{\Xi} \circ \Lambda = \mathrm{id}_{OF(Q,\mathcal{F}_Q)}$. Conversely, for $[e] \in OF(M, \mathcal{F}_M)/U(n)$ with $x := \pi(e)$, taking the choice $x \in p^{-1}(p(x))$ gives

$$(\Lambda \circ \overline{\Xi})([e]) = \Lambda(\Xi(e)) = \Lambda(\overline{dp}_x \circ e) = [\overline{dp}_x^{-1} \circ \overline{dp}_x \circ e] = [e],$$

hence $\Lambda \circ \overline{\Xi} = \mathrm{id}_{OF(M,\mathcal{F}_M)/U(n)}$. Thus $\overline{\Xi}$ is an orbifold diffeomorphism.

Finally, $\overline{\Xi}$ is $O(q)$-equivariant because Ξ is. And it covers the identity map on Q since $\pi_{\mathcal{O}} \circ \Xi = p \circ \pi$. Therefore $\overline{\Xi}$ is an isomorphism of orbifold principal $O(q)$-bundles.

(d) Let $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i \in I}$ be an orbifold atlas for Q. For each i, let $\mathcal{F}_i$ be the local foliation on U_i represented by a submersion $s_i : U_i \to T_i \subseteq \mathbb{R}^q$. Let $\pi_i : UF(U_i) \to U_i$ be the unitary frame bundle projection, and let $\mathcal{F}_i^M$ be the local pullback foliation on $UF(U_i)$ defined by

$\tilde{s}_i := s_i \circ \pi_i$. By Lemma 4.13, these local pullback foliations descend and glue to the global foliation $\mathcal{F}_M$ on $M = UF(Q)$.

Let $\pi_i^M : OF(UF(U_i), \mathcal{F}_i^M) \to UF(U_i)$ and $\pi_i^Q : OF(U_i, \mathcal{F}_i) \to U_i$ be the local orthonormal frame bundle projections. Define the local map

$$\Xi_i : OF(UF(U_i), \mathcal{F}_i^M) \to OF(U_i, \mathcal{F}_i), \qquad \Xi_i(\tilde{e}) := \overline{d\pi_i}_{\pi_i^M(\tilde{e})} \circ \tilde{e}.$$

Also define

$$\widehat{s}_i^M : OF(UF(U_i), \mathcal{F}_i^M) \to OF(T_i), \qquad \widehat{s}_i^M(\tilde{e}) := (d\tilde{s}_i)^N_{\pi_i^M(\tilde{e})} \circ \tilde{e},$$

and

$$\widehat{s}_i^Q : OF(U_i, \mathcal{F}_i) \to OF(T_i), \qquad \widehat{s}_i^Q(\bar{e}) := (ds_i)^N_{\pi_i^Q(\bar{e})} \circ \bar{e}.$$

These are the local defining maps for the lifted foliations on the two local frame bundles.

Now let $\tilde{e} \in OF(UF(U_i), \mathcal{F}_i^M)$ and let $\tilde{x} := \pi_i^M(\tilde{e})$. Since $\tilde{s}_i = s_i \circ \pi_i$, functoriality of the induced maps on normal bundles gives $(d\tilde{s}_i)^N_{\tilde{x}} = (ds_i)^N_{\pi_i(\tilde{x})} \circ \overline{d\pi_i}_{\tilde{x}}$. Therefore

$$\widehat{s}_i^Q(\Xi_i(\tilde{e})) = (ds_i)^N_{\pi_i(\tilde{x})} \circ \overline{d\pi_i}_{\tilde{x}} \circ \tilde{e} = (d\tilde{s}_i)^N_{\tilde{x}} \circ \tilde{e} = \widehat{s}_i^M(\tilde{e}).$$

All these constructions are G_i-equivariant. Since $q_i : UF(U_i) \to W_i := UF(U_i)/G_i$ is a local diffeomorphism and $\mathcal{F}_M|_{W_i}$ is the quotient of $\mathcal{F}_i^M$, the induced map on normal bundles identifies

$$OF(UF(U_i), \mathcal{F}_i^M)/G_i \cong OF(W_i, \mathcal{F}_M|_{W_i}).$$

Under this identification, the map induced by Ξ_i is the local representative of Ξ. Also, $\widehat{s}_i^Q \circ \Xi_i = \widehat{s}_i^M$. Hence the orbifold diffeomorphism

$$\overline{\Xi} : OF(M, \mathcal{F}_M)/U(n) \xrightarrow{\cong} OF(Q, \mathcal{F}_Q)$$

identifies the foliation descended from $\widetilde{\mathcal{F}}_M$ with the lifted foliation on $OF(Q, \mathcal{F}_Q)$. □

Lemma 4.17. *Let H act smoothly, properly, and locally freely on a manifold M, and let $\Pi : M \to R := M/H$ be the quotient map. Let $\mathcal{F}$ be a foliation on M such that the H-action is leafwise for $\mathcal{F}$.*

Let $\mathfrak{a}$ be a Lie algebra. If $\alpha \in \Omega^1(M; \mathfrak{a})$ is H-invariant and basic for $\mathcal{F}$, then there exists a unique orbifold 1-form $\bar{\alpha}$ on R such that $\Pi^*\bar{\alpha} = \alpha$. If α is a Maurer–Cartan form as in Definition 3.45, then $\bar{\alpha}$ satisfies the same equation on R.

Proof. Let $\mathfrak{h}$ be the Lie algebra of H, and for $\xi \in \mathfrak{h}$ let ξ_M be the fundamental vector field on M. Since the H-action is leafwise for $\mathcal{F}$, we have $\xi_M \in \Gamma(T\mathcal{F})$. Because α is basic for $\mathcal{F}$, it vanishes on vectors tangent to $\mathcal{F}$, hence $\alpha(\xi_M) = 0$ for all $\xi \in \mathfrak{h}$. Since α is H-invariant, we have $(\exp(t\xi))^*\alpha = \alpha$ for all t, and differentiating at $t = 0$ yields $L_{\xi_M}\alpha = 0$ for all $\xi \in \mathfrak{h}$. Thus α is H-horizontal and H-invariant.

Let $x \in M$. Since the action is proper and locally free, we may choose an H_x-invariant slice S_x through x and an H-equivariant diffeomorphism

$$\Phi_x : H \times_{H_x} S_x \xrightarrow{\cong} U_x, \qquad [h, s] \longmapsto h \cdot s,$$

onto an H-invariant neighbourhood U_x of the orbit $H \cdot x$. Because the action is locally free, H_x is discrete; because the action is proper, H_x is compact; hence H_x is finite.

Shrink S_x and choose local coordinates on it. We identify S_x with a connected open subset of $\mathbb{R}^n$. Then the restriction $\phi_x := \Pi|_{S_x} : S_x \to R$ is an open map inducing a homeomorphism $S_x/H_x \cong \Pi(U_x)$, so (S_x, H_x, ϕ_x) is an orbifold chart on $\Pi(U_x) \subseteq R$.

Let $\tilde{\alpha}_x := \alpha|_{S_x} \in \Omega^1(S_x; \mathfrak{a})$. Since α is H-invariant and S_x is H_x-stable, $\tilde{\alpha}_x$ is H_x-invariant. Hence there exists a unique orbifold 1-form $\bar{\alpha}_x$ on $\Pi(U_x)$ characterized by $\phi_x^*\bar{\alpha}_x = \tilde{\alpha}_x$.

Claim. On U_x we have $\alpha|_{U_x} = (\Pi|_{U_x})^*\bar{\alpha}_x$.

Consider the action map $\varphi : H \times S_x \to U_x$, $\varphi(h, s) = h \cdot s$, and let $\mathrm{pr}_{S_x} : H \times S_x \to S_x$ be the projection. Since $\Pi(h \cdot s) = \Pi(s)$, we have $(\Pi|_{U_x}) \circ \varphi = \phi_x \circ \mathrm{pr}_{S_x}$.

We compute $\varphi^*\alpha$. If $(v, 0) \in T_hH \oplus 0$, then $d\varphi_{(h,s)}(v, 0)$ is tangent to the H-orbit through $h \cdot s$; by H-horizontality of α, this gives $(\varphi^*\alpha)_{(h,s)}(v, 0) = 0$. If $(0, \eta) \in 0 \oplus T_sS_x$, then $d\varphi_{(h,s)}(0, \eta) = d(\varphi_h)_s(\eta)$, and H-invariance gives

$$(\varphi^*\alpha)_{(h,s)}(0, \eta) = \alpha_{h\cdot s}(d(\varphi_h)_s\eta) = (\varphi_h^*\alpha)_s(\eta) = \alpha_s(\eta) = (\tilde{\alpha}_x)_s(\eta).$$

Hence $\varphi^*\alpha = \mathrm{pr}_{S_x}^*\,\tilde{\alpha}_x$. On the other hand,

$$\varphi^*(\Pi|_{U_x})^*\bar{\alpha}_x = (\phi_x \circ \mathrm{pr}_{S_x})^*\bar{\alpha}_x = \mathrm{pr}_{S_x}^*\,\phi_x^*\bar{\alpha}_x = \mathrm{pr}_{S_x}^*\,\tilde{\alpha}_x.$$

Therefore $\varphi^*\big(\alpha - (\Pi|_{U_x})^*\bar{\alpha}_x\big) = 0$. Since φ is a surjective submersion, it admits local smooth sections, so pullback by φ is injective on 1-forms. This proves the claim.

On an overlap $U_x \cap U_{x'}$ we have $(\Pi|_{U_x\cap U_{x'}})^*\bar{\alpha}_x = \alpha = (\Pi|_{U_x\cap U_{x'}})^*\bar{\alpha}_{x'}$. Pullback along Π is injective on orbifold forms: if an orbifold form β on R satisfies $\Pi^*\beta = 0$, then on any slice chart (S_x, H_x, ϕ_x) we have $\phi_x^*\beta = (\Pi|_{S_x})^*\beta = 0$, hence $\beta = 0$. Therefore $\bar{\alpha}_x = \bar{\alpha}_{x'}$ on

$\Pi(U_x \cap U_{x'})$, and the $\bar{\alpha}_x$ glue to a global orbifold 1-form $\bar{\alpha}$ on R satisfying $\Pi^*\bar{\alpha} = \alpha$. The same injectivity gives uniqueness.

Orbifold pullback is defined chartwise and hence commutes with d and with the bracket. Therefore

$$\Pi^*\Big(d\bar{\alpha} + \frac{1}{2}[\bar{\alpha}, \bar{\alpha}]\Big) = d(\Pi^*\bar{\alpha}) + \frac{1}{2}[\Pi^*\bar{\alpha}, \Pi^*\bar{\alpha}] = d\alpha + \frac{1}{2}[\alpha, \alpha].$$

If the right-hand side vanishes, then the left-hand side vanishes. By injectivity of Π^*, we conclude $d\bar{\alpha} + \frac{1}{2}[\bar{\alpha}, \bar{\alpha}] = 0$ on R. □

Proof of Corollary 4.10

Remark 4.18. Recall Definition 4.9. Let $\{(U_i, G_i, \phi_i)\}_{i\in I}$ be the orbifold atlas of Q, and let $\mathcal{F}_i$ be the local foliations representing $\mathcal{F}_Q$. We say that $\mathcal{F}_Q$ is a Lie foliation defined by an orbifold $\mathfrak{g}$-valued Maurer–Cartan form if there exist G_i-invariant 1-forms $\eta_i \in \Omega^1(U_i; \mathfrak{g})$ such that for every embedding of orbifold charts $\lambda : (U_i, G_i, \phi_i) \to (U_j, G_j, \phi_j)$, we have $\lambda^*\eta_j = \eta_i$, and each η_i is pointwise surjective, satisfies $T(\mathcal{F}_i) = \ker(\eta_i)$, and $d\eta_i + \frac{1}{2}[\eta_i, \eta_i] = 0$.

We now have enough tools to give a proof of the main statement.

Proof. Let $p : M := UF(Q) \to Q$ be the unitary frame bundle of the orbifold. By Lemmas 4.11 and 4.13, M is a compact connected manifold (since Q is), equipped with the pullback Riemannian foliation $(\mathcal{F}_M, g_M^T)$, and the $U(n)$-action on M is fibrewise with respect to p and acts by foliated isometries of $(\mathcal{F}_M, g_M^T)$. Since $U(n)$ is connected, Proposition 4.14 implies that $U(n)$ acts leafwise on $\mathcal{F}_M$ and, after lifting, leafwise on the lifted foliation $\widetilde{\mathcal{F}}_M$ on $OF(M, \mathcal{F}_M)$.

Apply Molino's structure theorem 2.4 to $(\mathcal{F}_M, g_M^T)$. Let $s_M : OF(M, \mathcal{F}_M) \to \mathcal{N}$ be the $O(q)$-equivariant Molino fibration, let θ_M be the transverse canonical form on $OF(M, \mathcal{F}_M)$, and let ω_M be the transverse Levi–Civita connection form on $OF(M, \mathcal{F}_M)$. By Proposition 4.7, the lifted $U(n)$-action on $OF(M, \mathcal{F}_M)$ preserves $\widetilde{\mathcal{F}}_M$, θ_M, and ω_M. Since the $U(n)$-action is leafwise for $\widetilde{\mathcal{F}}_M$, it preserves closures of leaves. Hence s_M is $U(n)$-invariant.

Let $\Pi : OF(M, \mathcal{F}_M) \to OF(M, \mathcal{F}_M)/U(n)$ be the quotient map.

By Definition 4.9 and Lemma 4.16, we identify $OF(M, \mathcal{F}_M)/U(n)$ with the orbifold transverse orthonormal frame bundle $\mathcal{O} := OF(Q, \mathcal{F}_Q)$, in such a way that the descended foliation is identified with the lifted foliation $\widetilde{\mathcal{F}}$ on $\mathcal{O}$. Under this identification, s_M factors uniquely as $s_M = \bar{s} \circ \Pi$ for an $O(q)$-equivariant orbifold map $\bar{s} : \mathcal{O} \to \mathcal{N}$.

Since $s_M : OF(M, \mathcal{F}_M) \to \mathcal{N}$ is a $U(n)$-invariant smooth fibre bundle and the $U(n)$-action on $OF(M, \mathcal{F}_M)$ is proper, locally free, and fibrewise with respect to s_M, local trivializations of s_M may be chosen to descend to quotient charts as local orbifold trivializations of $\bar{s}$. Hence $\bar{s} : \mathcal{O} \to \mathcal{N}$ is an $O(q)$-equivariant orbifold fibre bundle.

(i) The form θ_M is basic for the foliation $\widetilde{\mathcal{F}}_M$ by its local pullback description from the transverse frame bundle construction, and ω_M is basic because it is a projectable connection form. Since the lifted $U(n)$-action is proper, locally free, leafwise for $\widetilde{\mathcal{F}}_M$, and preserves θ_M and ω_M, Lemma 4.17 yields unique orbifold 1-forms θ, ω on $\mathcal{O}$ such that $\Pi^*\theta = \theta_M$ and $\Pi^*\omega = \omega_M$. These are the orbifold transverse canonical form and the orbifold transverse Levi–Civita connection form on $\mathcal{O}$.

Since $T(\widetilde{\mathcal{F}}_M) = \ker \theta_M \cap \ker \omega_M$ on $OF(M, \mathcal{F}_M)$, the same equality holds chartwise on $\mathcal{O}$: $T(\widetilde{\mathcal{F}}) = \ker \theta \cap \ker \omega$. Therefore $(\mathcal{O}, \widetilde{\mathcal{F}})$ is transversely parallelizable, and the pair (θ, ω) determines the corresponding transverse parallelism.

(ii) The map s_M descends to $\bar{s}$ as discussed above. We first identify the fibres of $\bar{s}$. Let $y \in \mathcal{N}$, and let L be a leaf of $\widetilde{\mathcal{F}}_M$ with $\overline{L} = s_M^{-1}(y)$. By Lemma 4.16(d), $\Pi(L)$ is a leaf of the descended foliation, identified with a leaf of $\widetilde{\mathcal{F}}$ on $\mathcal{O}$. Hence Lemma 4.15 gives $\bar{s}^{-1}(y) = \Pi\big(s_M^{-1}(y)\big) = \Pi(\overline{L}) = \overline{\Pi(L)}$. Thus the fibres of $\bar{s}$ are the closures of the leaves of $\widetilde{\mathcal{F}}$.

(iii) Fix $y \in \mathcal{N}$, and let $X_y := s_M^{-1}(y)$, $\bar{X}_y := \bar{s}^{-1}(y) = X_y/U(n)$. By Theorem 2.4, the restricted foliation $\widetilde{\mathcal{F}}_M|_{X_y}$ is a Lie foliation defined by a canonical Maurer–Cartan form with values in the Lie algebra $\mathfrak{g}_y := l\big(X_y, \widetilde{\mathcal{F}}_M|_{X_y}\big)$, and the Lie algebras $\mathfrak{g}_y$ are independent of y up to isomorphism.

Choose a Lie algebra $\mathfrak{g}$ together with isomorphisms $\mathfrak{g}_y \xrightarrow{\cong} \mathfrak{g}$, and regard the canonical Maurer–Cartan form as a $\mathfrak{g}$-valued 1-form $\omega^y_{\mathrm{MC}} \in \Omega^1(X_y; \mathfrak{g})$. It satisfies

$$T(\widetilde{\mathcal{F}}_M|_{X_y}) = \ker(\omega^y_{\mathrm{MC}}), \qquad d\omega^y_{\mathrm{MC}} + \tfrac{1}{2}[\omega^y_{\mathrm{MC}}, \omega^y_{\mathrm{MC}}] = 0.$$

For any $Y \in \Gamma\big(T(\widetilde{\mathcal{F}}_M|_{X_y})\big)$ we have $\iota_Y \omega^y_{\mathrm{MC}} = 0$. Using Cartan's formula,

$$L_Y \omega^y_{\mathrm{MC}} = \iota_Y(d\omega^y_{\mathrm{MC}}) + d(\iota_Y \omega^y_{\mathrm{MC}}) = -\tfrac{1}{2}\,\iota_Y[\omega^y_{\mathrm{MC}}, \omega^y_{\mathrm{MC}}] = 0,$$

so ω^y_{MC} is basic for the foliation $\widetilde{\mathcal{F}}_M|_{X_y}$.

Now let $\xi \in \mathfrak{u}(n)$, and let $\xi^\#$ be the corresponding fundamental vector field on X_y. Since the $U(n)$-action on X_y is leafwise for $\widetilde{\mathcal{F}}_M|_{X_y}$, we have $\xi^\# \in \Gamma\big(T(\widetilde{\mathcal{F}}_M|_{X_y})\big)$, hence $L_{\xi^\#}\omega^y_{\mathrm{MC}} = 0$. Because $U(n)$ is connected, it follows that ω^y_{MC} is $U(n)$-invariant.

Let $\Pi_y : X_y \to \bar{X}_y$ be the quotient map. Applying Lemma 4.17 to the $U(n)$-action on X_y and the $U(n)$-invariant form ω^y_{MC}, we obtain a unique orbifold 1-form $\bar{\omega}^y_{\mathrm{MC}} \in \Omega^1(\bar{X}_y; \mathfrak{g})$ such that $\Pi_y^* \bar{\omega}^y_{\mathrm{MC}} = \omega^y_{\mathrm{MC}}$. It is again a Maurer–Cartan form. Since Π_y is a submersion and ω^y_{MC} is pointwise surjective, $\bar{\omega}^y_{\mathrm{MC}}$ is pointwise surjective as well.

Therefore $\bar{X}_y$ carries a Lie foliation, which is the restriction $\widetilde{\mathcal{F}}|_{\bar{X}_y}$, with $T(\widetilde{\mathcal{F}}|_{\bar{X}_y}) = \ker(\bar{\omega}^y_{\mathrm{MC}})$.

It remains to prove dense holonomy. Let $G_{\mathfrak{g}}$ be the connected and simply connected Lie group integrating $\mathfrak{g}$. Let η^y and $\bar{\eta}^y$ be the Darboux connection forms on $X_y \times G_{\mathfrak{g}}$ and $\bar{X}_y \times G_{\mathfrak{g}}$ associated to ω_{MC}^y and $\bar{\omega}_{\mathrm{MC}}^y$, respectively. By functoriality of the construction,

$$(\Pi_y \times \mathrm{id}_{G_{\mathfrak{g}}})^* \bar{\eta}^y = \eta^y.$$

Choose $x_0 \in X_y$, let $\widetilde{X}_y \subseteq X_y \times G_{\mathfrak{g}}$ be the leaf of $\ker(\eta^y)$ through $(x_0, 1)$, and let $\widetilde{\bar{X}}_y \subseteq \bar{X}_y \times G_{\mathfrak{g}}$ be the leaf of $\ker(\bar{\eta}^y)$ through $(\Pi_y(x_0), 1)$. Then $(\Pi_y \times \mathrm{id}_{G_{\mathfrak{g}}})(\widetilde{X}_y) \subseteq \widetilde{\bar{X}}_y$. Let

$$H_y := \{g \in G_{\mathfrak{g}} \mid \widetilde{X}_y \cdot g = \widetilde{X}_y\}, \qquad \bar{H}_y := \{g \in G_{\mathfrak{g}} \mid \widetilde{\bar{X}}_y \cdot g = \widetilde{\bar{X}}_y\}.$$

If $g \in H_y$, then $\widetilde{X}_y \cdot g = \widetilde{X}_y$, hence

$$(\Pi_y \times \mathrm{id}_{G_{\mathfrak{g}}})(\widetilde{X}_y) \cdot g = (\Pi_y \times \mathrm{id}_{G_{\mathfrak{g}}})(\widetilde{X}_y \cdot g) = (\Pi_y \times \mathrm{id}_{G_{\mathfrak{g}}})(\widetilde{X}_y) \subseteq \widetilde{\bar{X}}_y.$$

Since the right $G_{\mathfrak{g}}$-action preserves $\ker(\bar{\eta}^y)$, the translate $\widetilde{\bar{X}}_y \cdot g$ is again a leaf of $\ker(\bar{\eta}^y)$. As it intersects $\widetilde{\bar{X}}_y$, we must have $\widetilde{\bar{X}}_y \cdot g = \widetilde{\bar{X}}_y$. Thus $g \in \bar{H}_y$, and so $H_y \subseteq \bar{H}_y$. By Molino's structure theorem on X_y, the holonomy group H_y is dense in $G_{\mathfrak{g}}$. Therefore $\bar{H}_y$ is dense in $G_{\mathfrak{g}}$ as well. Hence the Lie foliation on $\bar{X}_y = \bar{s}^{-1}(y)$ has dense holonomy.

Finally, because each $\mathfrak{g}_y$ is isomorphic to $\mathfrak{g}$, the structural Lie algebra of the Lie foliation on $\bar{s}^{-1}(y)$ is independent of y up to isomorphism. □

4.3 Proper étale groupoid case

We proceed in two parts. In the first part, we build the orbifold framework associated to a proper étale groupoid and a G-invariant Riemannian foliation. In Part II, we apply Corollary 4.10 and transport the resulting structure, giving our main proof of Corollary 4.38.

For the remainder of this subsection, we assume that $G \rightrightarrows M$ is Hausdorff, proper, effective, and étale. Following [12, Sec. 5.5], the source and target maps $s, t : G_1 \to M$ are local diffeomorphisms and $(s, t) : G_1 \to M \times M$ is proper. In particular, all isotropy groups G_x are finite.

From the groupoid to the orbifold frame bundle

Remark 4.19. Let $G \rightrightarrows M$ be a proper effective étale Lie groupoid. By [12, Prop. 5.30], for each $x \in M$ there exists an open neighbourhood $U \subseteq M$ and an action of the finite isotropy group G_x on U such that $G|_U \cong G_x \ltimes U$ as étale Lie groupoids. Since G is effective, the quotients U/G_x are effective local orbifold charts on $Q := M/G$, and by [12, Cor. 5.31] they determine the canonical orbifold structure on Q.

To avoid confusion with the Haefliger groupoid of germs $\Gamma(M)$, we denote by $\mathcal{G}_Q^{\text{eff}} \rightrightarrows Q$ the associated proper effective orbifold groupoid. Then $\mathcal{G}_Q^{\text{eff}}$ is weakly equivalent to $\text{Eff}(G)$.

Remark 4.20. ([12, Sec. 5.5]) Let $G \rightrightarrows M$ be an étale Lie groupoid. There is a canonical homomorphism of Lie groupoids

$$\text{Eff} : G \longrightarrow \Gamma(M),$$

called the *effect homomorphism*. The étale groupoid G is called *effective* if Eff is injective on arrows. The image $\mathrm{Eff}(G) \subseteq \Gamma(M)$ is an open subgroupoid (hence effective), called the *effect* of G.

Definition 4.21. Let $\gamma \in G_1$ with $s(\gamma) = x$ and $t(\gamma) = y$. Since s and t are local diffeomorphisms, there exist open neighbourhoods $U \subseteq M$ of x and $W \subseteq G_1$ of γ, and a diffeomorphism (a local bisection through γ) $\sigma : U \xrightarrow{\cong} W$ such that $\sigma(x) = \gamma$, $s \circ \sigma = \mathrm{id}_U$, and $t \circ \sigma$ is an open embedding. This allows us to set

$$\varphi_\gamma := t \circ \sigma : U \longrightarrow M.$$

Lemma 4.22. *Let $\sigma_1, \sigma_2 : U \to G_1$ be local bisections with $s \circ \sigma_i = \mathrm{id}_U$ and $\sigma_i(x) = \gamma$. Then, after shrinking U around x if necessary, we have $\sigma_1 = \sigma_2$ on U. Hence $\varphi_\gamma = t \circ \sigma_1 = t \circ \sigma_2$ on U. In particular, the germ of φ_γ at x is well-defined, independent of the choice of σ.*

Proof. Since s is a local diffeomorphism, there exists a neighbourhood $W \subseteq G_1$ of γ such that $s|_W : W \xrightarrow{\cong} s(W)$ is a diffeomorphism. After shrinking U, we may assume that $U \subseteq s(W)$ and $\sigma_i(U) \subseteq W$ for $i = 1, 2$. Then $s|_W \circ \sigma_i = s \circ \sigma_i = \mathrm{id}_U$, so $\sigma_i = (s|_W)^{-1}|_U$ for $i = 1, 2$. Hence $\sigma_1 = \sigma_2$ on U. □

Following Lemma 4.22, each arrow $\gamma : x \to y$ determines a well-defined germ $[\varphi_\gamma] \in \Gamma(M)(x, y)$. Lemma 4.23 shows that these germs respect composition, and hence they assemble to the effect homomorphism $\mathrm{Eff} : G \to \Gamma(M)$ from Remark 4.20.

Lemma 4.23. *Let $\gamma : x \to y$ and $\eta : y \to z$ be arrows in G. Then, as germs at x, $\varphi_{\eta\gamma} = \varphi_\eta \circ \varphi_\gamma$.*

Proof. Choose local bisections $\sigma_\gamma : U_\gamma \to G_1$ with $\sigma_\gamma(x) = \gamma$ and $\sigma_\eta : U_\eta \to G_1$ with $\sigma_\eta(y) = \eta$. After shrinking U_γ, assume that $\varphi_\gamma(U_\gamma) \subseteq U_\eta$, where $\varphi_\gamma = t \circ \sigma_\gamma$ and $\varphi_\eta = t \circ \sigma_\eta$. Define $\sigma_{\eta\gamma}(u) := \sigma_\eta(\varphi_\gamma(u))\sigma_\gamma(u)$ for $u \in U_\gamma$. Then $s(\sigma_{\eta\gamma}(u)) = s(\sigma_\gamma(u)) = u$, $\sigma_{\eta\gamma}(x) = \eta\gamma$, and

$$t \circ \sigma_{\eta\gamma}(u) = t\big(\sigma_\eta(\varphi_\gamma(u))\sigma_\gamma(u)\big) = t\big(\sigma_\eta(\varphi_\gamma(u))\big) = \varphi_\eta(\varphi_\gamma(u))$$

for all $u \in U_\gamma$. Hence $\sigma_{\eta\gamma}$ is a local bisection through $\eta\gamma$, and $\varphi_{\eta\gamma} = t \circ \sigma_{\eta\gamma} = \varphi_\eta \circ \varphi_\gamma$ as germs at x. □

Now we want to discuss constructing G-invariant Riemannian foliations and descent to $Q = M/G$. Let $(\mathcal{F}, g)$ be a Riemannian foliation on M, where g is a transverse metric.

Definition 4.24. We say that $(\mathcal{F}, g)$ is G-invariant if for every arrow $\gamma : x \to y$ the germ $\mathrm{Eff}(\gamma) \in \Gamma(M)(x, y)$ is represented by a local diffeomorphism φ_γ which is a foliated isometry of $(\mathcal{F}, g)$, i.e. it sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$ and is an isometry for the transverse metric g.

Equivalently, for every arrow γ the germ $\mathrm{Eff}(\gamma) \in \Gamma(M)(x, y)$ is represented by a local foliated isometry.

Corollary 4.25. *Assume $(\mathcal{F}, g)$ is G-invariant. For each arrow $\gamma : x \to y$, the induced local foliated isometry φ_γ yields a well-defined local diffeomorphism*

$$OF(\varphi_\gamma) : OF(M, \mathcal{F})|_{\mathrm{dom}(\varphi_\gamma)} \to OF(M, \mathcal{F})|_{\mathrm{im}(\varphi_\gamma)}$$

which preserves the lifted foliation $\widetilde{\mathcal{F}}$ and satisfies

$$OF(\varphi_\gamma)^*\theta = \theta, \qquad OF(\varphi_\gamma)^*\omega = \omega,$$

on the common domain.

Proof. By G-invariance, φ_γ is a foliated transverse isometry. Applying Lemma 3.23 completes the proof. $\square$

Lemma 4.26. *Let Γ be a finite group acting smoothly and effectively on a manifold U. Let $(\mathcal{F}, g)$ be a Riemannian foliation on U, and assume that every $\gamma \in \Gamma$ acts by a foliated isometry of $(\mathcal{F}, g)$.*

Then the quotient orbifold U/Γ carries a unique Riemannian foliation $(\mathcal{F}_{U/\Gamma}, \bar{g}_{U/\Gamma})$ such that, for the quotient map $p_\Gamma : U \to U/\Gamma$, we have

$$\mathcal{F} = p_\Gamma^{-1}(\mathcal{F}_{U/\Gamma}), \qquad g = p_\Gamma^* \bar{g}_{U/\Gamma}$$

as transverse metrics.

Proof. Step 1: We choose a Haefliger cocycle

Choose a Haefliger cocycle (U_i, s_i, γ_{ij}) for $\mathcal{F}$ with each $s_i : U_i \to T_i \subseteq \mathbb{R}^q$ a submersion with connected fibres, and choose it so that g is induced from Riemannian metrics g_i on the T_i, with the transition maps γ_{ij} local isometries.

Step 2: Produce the induced map $\bar{\delta}_i$ and show it is a local isometry

Since Γ is finite, after refining the cover, we may assume that for every i and $\delta \in \Gamma$, $\delta(U_i) \subseteq U_j$. Because the fibres of $s_i : U_i \to T_i$ are connected plaques of $\mathcal{F}|_{U_i}$ and δ is foliated, $s_j \circ \delta : U_i \to T_j$ is constant on the fibres of s_i. If $s_i(x) = s_i(x')$, then x and x' lie in the same connected fibre of s_i, hence in the same plaque of $\mathcal{F}|_{U_i}$, and then $\delta(x)$ and $\delta(x')$ lie in the same plaque of $\mathcal{F}|_{U_j}$, so $s_j(\delta(x)) = s_j(\delta(x'))$.

It follows that there is a unique map $\bar{\delta}_i : s_i(U_i) \to s_j(\delta(U_i))$ such that $s_j \circ \delta = \bar{\delta}_i \circ s_i$ on U_i. Since s_i is a surjective submersion onto $s_i(U_i)$, $\bar{\delta}_i$ is smooth. We want $\bar{\delta}_i$ to be a diffeomorphism, so we need to show that its inverse is smooth. Since δ^{-1} is also a foliated diffeomorphism, there is a unique smooth map $\overline{\delta^{-1}}_j : s_j(\delta(U_i)) \to s_i(U_i)$ such that $s_i \circ \delta^{-1} = \overline{\delta^{-1}}_j \circ s_j$ on $\delta(U_i)$. Then we compute $s_i = s_i \circ \delta^{-1} \circ \delta = \overline{\delta^{-1}}_j \circ s_j \circ \delta = \overline{\delta^{-1}}_j \circ \bar{\delta}_i \circ s_i$. Because s_i is surjective onto $s_i(U_i)$, it follows that $\overline{\delta^{-1}}_j \circ \bar{\delta}_i = \mathrm{id}_{s_i(U_i)}$. Similarly, $\bar{\delta}_i \circ \overline{\delta^{-1}}_j = \mathrm{id}_{s_j(\delta(U_i))}$. So $\bar{\delta}_i$ is a diffeomorphism onto its image.

Let g_i and g_j be Riemannian metrics on T_i and T_j representing the transverse metric g on U_i and U_j. Because δ is a foliated isometry of $(\mathcal{F}, g)$, $\delta^*(g|_{\delta(U_i)}) = g|_{U_i}$. On the cocycle charts, $g|_{U_i} = s_i^* g_i$ and $g|_{U_j} = s_j^* g_j$. Using $s_j \circ \delta = \bar{\delta}_i \circ s_i$, we have $s_i^*(\bar{\delta}_i^* g_j) = s_i^* g_i$. Again, since s_i is a surjective submersion, $\bar{\delta}_i^* g_j = g_i$ on $s_i(U_i)$.

For now we can say that, on each local quotient chart from the finite group action, the local foliation defined by s_i and the local transverse metric g_i are invariant under the action of the finite subgroup of Γ preserving that chart, and hence descend to the chart quotient.

Step 3: Local foliations glue

On overlaps of quotient charts, the change-of-charts maps are induced by elements of Γ. By $s_j \circ \delta = \bar{\delta}_i \circ s_i$, these maps identify the descended local submersions, while $\bar{\delta}_i^* g_j = g_i$ shows that they preserve the descended transverse metric. Therefore these local quotient data glue to a Riemannian foliation $(\mathcal{F}_{U/\Gamma}, \bar{g}_{U/\Gamma})$ on U/Γ.

Step 4: Pullback identities and uniqueness

By construction, on each local chart V/H, the downstairs foliation was defined so that its pullback to V is $\mathcal{F}|_V$. So $\mathcal{F}|_V = p_\Gamma^{-1}(\mathcal{F}_{U/\Gamma})|_V$. They glue to $\mathcal{F} = p_\Gamma^{-1}(\mathcal{F}_{U/\Gamma})$. Likewise for the downstairs transverse metric. Locally $g|_V = p_\Gamma^*(\bar{g}_{U/\Gamma})|_V$, and they glue to the global identity $g = p_\Gamma^* \bar{g}_{U/\Gamma}$.

Uniqueness follows because on each manifold chart covering the orbifold, the downstairs foliation and transverse metric are uniquely determined by their pullback upstairs, and the orbifold structure on U/Γ is defined by the quotient charts. $\square$

Proposition 4.27. *Let $G \rightrightarrows M$ be a proper effective étale groupoid, and let $(\mathcal{F}, g)$ be G-invariant on M. Then $Q := M/G$ carries a Riemannian foliation $(\mathcal{F}_Q, \bar{g})$ such that, for the quotient map $p : M \to Q$,*

$$\mathcal{F} = p^{-1}(\mathcal{F}_Q), \qquad g = p^*\bar{g}$$

as transverse metrics.

Proof. Let $x \in M$. By Remark 4.19, choose an open neighbourhood $U \subseteq M$ of x with $G|_U \cong G_x \ltimes U$. Since $(\mathcal{F}, g)$ is G-invariant, every element of the finite isotropy group G_x acts on U by a foliated isometry of $(\mathcal{F}|_U, g|_U)$.

Descend:

By Lemma 4.26, the quotient orbifold U/G_x carries a unique Riemannian foliation $(\mathcal{F}_{U/G_x}, \bar{g}_{U/G_x})$ whose pullback along the quotient map $p_U : U \to U/G_x$ is $(\mathcal{F}|_U, g|_U)$.

Glue the local quotient foliations on Q:

Now let U/G_x and $U'/G_{x'}$ be two such local quotient charts on $Q = M/G$. Every change-of-charts map between them is induced by a germ of an arrow of G. Since $(\mathcal{F}, g)$ is G-invariant, that germ is represented by a local foliated isometry upstairs. By the same reasoning as in Lemma 4.26, the local Riemannian foliations $(\mathcal{F}_{U/G_x}, \bar{g}_{U/G_x})$ glue to a global Riemannian foliation $(\mathcal{F}_Q, \bar{g})$ on the orbifold Q.

By construction, on each local quotient chart U/G_x the quotient map p_U pulls back $(\mathcal{F}_{U/G_x}, \bar{g}_{U/G_x})$ to $(\mathcal{F}|_U, g|_U)$. Hence globally $\mathcal{F} = p^{-1}(\mathcal{F}_Q)$, $g = p^*\bar{g}$. □

For $x \in M$ set $N_x(\mathcal{F}) := T_xM/T_x(\mathcal{F})$. If φ is a local foliated diffeomorphism defined on a neighbourhood of x, then $d\varphi_x$ preserves $T_x(\mathcal{F})$ and induces a linear isomorphism $(d\varphi)^N(x) : N_x(\mathcal{F}) \xrightarrow{\cong} N_{\varphi(x)}(\mathcal{F})$. In addition, if φ is a transverse isometry for g, then $(d\varphi)^N(x)$ is orthogonal. In particular, for an arrow $\gamma : x \to y$ we write

$$(d\varphi)^N_\gamma(x) := (d\varphi_\gamma)^N(x) : N_x(\mathcal{F}) \to N_y(\mathcal{F}),$$

where φ_γ is any representative of the germ $\mathrm{Eff}(\gamma)$ near x.

Lemma 4.28. *The linear map $(d\varphi)^N_\gamma(x)$ depends only on the arrow γ and not on the choice of local bisection used to define φ_γ.*

Proof. By Lemma 4.22, the germ of φ_γ at x is independent of the chosen local bisection. Hence, $d\varphi_{\gamma,x} : T_xM \to T_yM$ is well-defined, and because φ_γ is foliated, it preserves $T_x(\mathcal{F})$. Therefore the induced map $(d\varphi_\gamma)^N_x : N_x(\mathcal{F}) \to N_y(\mathcal{F})$ is well-defined as well. □

Thus, each isotropy group G_x acts orthogonally on $N_x(\mathcal{F})$, and more generally G acts functorially on $N(\mathcal{F})$ by fibrewise linear isometries.

Let $q := \operatorname{codim} \mathcal{F}$, and let $\pi : OF(M, \mathcal{F}) \to M$ be the transverse orthonormal frame bundle. We use $\widetilde{\mathcal{F}}$ for the lifted foliation, and θ, ω for the transverse canonical form and transverse Levi–Civita connection form, as in Definitions 3.15, 3.17, and 3.19.

Now let $\gamma : x \to y$ be an arrow of G. For $e \in OF_x(M, \mathcal{F})$, we define

$$\gamma \cdot e := (d\varphi)^N_\gamma(x) \circ e \in OF_y(M, \mathcal{F}). \tag{4.29}$$

Lemma 4.30. *The map $\gamma \mapsto \gamma \cdot (-)$ defines a smooth left action of G on $OF(M, \mathcal{F})$.*

(a) If 1_x is the unit arrow at $x \in M$, then $1_x \cdot e = e$ for all $e \in OF_x(M, \mathcal{F})$.

(b) If $\gamma : x \to y$ and $\eta : y \to z$, then $(\eta\gamma) \cdot e = \eta \cdot (\gamma \cdot e)$ for all $e \in OF_x(M, \mathcal{F})$.

Moreover, the action commutes with the right $O(q)$-action: $\gamma \cdot (eA) = (\gamma \cdot e)A$ for all $A \in O(q)$.

Proof. We first show smoothness. Consider the action map

$$\mu : G_1 \times_M OF(M, \mathcal{F}) \longrightarrow OF(M, \mathcal{F}), \qquad (\gamma, e) \longmapsto \gamma \cdot e.$$

Locally, let (γ_0, e_0) with $x_0 := \pi(e_0) = s(\gamma_0)$. Choose a local bisection $\sigma : U \to G_1$ with $\sigma(x_0) = \gamma_0$ and let $\varphi := t \circ \sigma : U \to V$. Then $\varphi : U \xrightarrow{\cong} V$ is a diffeomorphism onto the open subset $V \subseteq M$.

On U, the subset $\{(\gamma, e) \mid \gamma \in \sigma(U),\ s(\gamma) = \pi(e)\} \subseteq G_1 \times_M OF(M, \mathcal{F})$ identifies smoothly with $OF(M, \mathcal{F})|_U$ via $(\sigma(\pi(e)), e) \leftrightarrow e$. In these coordinates, the action is $e \longmapsto (d\varphi)^N(\pi(e)) \circ e$, which is smooth. Hence μ is smooth.

(a) For $1_x : x \to x$ we may take $\sigma(u) = 1_u$ on a neighbourhood of x. Then $\varphi_{1_x} = \mathrm{id}$ near x and $(d\varphi)^N_{1_x}(x) = \mathrm{id}_{N_x(\mathcal{F})}$, so $1_x \cdot e = e$.

(b) Let $\gamma : x \to y$ and $\eta : y \to z$. By Lemma 4.23, $\varphi_{\eta\gamma} = \varphi_\eta \circ \varphi_\gamma$ as germs at x, hence

$$(d\varphi)^N_{\eta\gamma}(x) = (d\varphi)^N_\eta(y) \circ (d\varphi)^N_\gamma(x).$$

Therefore $(\eta\gamma) \cdot e = \eta \cdot (\gamma \cdot e)$.

Lastly, it is immediate from (4.29) that $\gamma \cdot (eA) = (d\varphi)^N_\gamma(x) \circ e \circ A = (\gamma \cdot e)A$. $\square$

Now we consider the action groupoid and its properties. Let

$$H := G \ltimes OF(M, \mathcal{F})$$

be the translation groupoid. Thus $H_0 = OF(M, \mathcal{F})$ and $H_1 = G_1 \times_M OF(M, \mathcal{F}) = \{(\gamma, e) \in G_1 \times OF(M, \mathcal{F}) \mid s(\gamma) = \pi(e)\}$, with $s_H(\gamma, e) = e$ and $t_H(\gamma, e) = \gamma \cdot e$.

Lemma 4.31 (Etaleness)**.** *The translation groupoid H is étale: $s_H, t_H : H_1 \to H_0$ are local diffeomorphisms.*

Proof. Let $(\gamma_0, e_0) \in H_1$ with $x_0 = \pi(e_0) = s(\gamma_0)$. Choose a local bisection $\sigma : U \to G_1$ with $\sigma(x_0) = \gamma_0$ and set $\varphi = t \circ \sigma : U \to V = \varphi(U)$. Then $\varphi : U \xrightarrow{\cong} V$ is a diffeomorphism onto the open subset $V \subseteq M$.

On U, the subset $H_1|_{\sigma(U)} := \{(\gamma, e) \in H_1 \mid \gamma \in \sigma(U)\}$ identifies with $OF(M, \mathcal{F})|_U$ via

$$\Psi : OF(M, \mathcal{F})|_U \xrightarrow{\cong} H_1|_{\sigma(U)}, \qquad e \longmapsto (\sigma(\pi(e)), e).$$

In these coordinates, $s_H \circ \Psi = \mathrm{id}$, so s_H is a local diffeomorphism.

Moreover,

$$t_H \circ \Psi(e) = \sigma(\pi(e)) \cdot e = (d\varphi)^N(\pi(e)) \circ e,$$

which is the induced map $OF(\varphi) : OF(M, \mathcal{F})|_U \to OF(M, \mathcal{F})|_V$. Since φ is a diffeomorphism, $OF(\varphi)$ is a diffeomorphism (with inverse $OF(\varphi^{-1})$). Therefore t_H is a local diffeomorphism. $\square$

Lemma 4.32 ([6, Chapter 10, Problem 10.19(c)])**.** *Let $f : E \to B$ be a locally trivial smooth fibre bundle with compact fibre F, and assume that B is locally compact Hausdorff. Then f is a proper map.*

Lemma 4.33 (Properness)**.** *The translation groupoid H is proper: $(s_H, t_H) : H_1 \to OF(M,\mathcal{F}) \times OF(M,\mathcal{F})$ is proper.*

Proof. By Lemma 4.32, the bundle projection $\pi : OF(M,\mathcal{F}) \to M$ is proper (since it is locally trivial with compact fibre $O(q)$ and M is locally compact Hausdorff). Therefore $\pi \times \pi$ is proper.

Let $\mathrm{pr}_1 : H_1 = G_1 \times_M OF(M,\mathcal{F}) \to G_1$ be the projection $(\gamma, e) \mapsto \gamma$. Since π is locally trivial with fibre $O(q)$, the pullback pr_1 is also locally trivial with fibre $O(q)$, hence proper by Lemma 4.32.

We have the identity

$$(s,t) \circ \mathrm{pr}_1 = (\pi \times \pi) \circ (s_H, t_H) : H_1 \longrightarrow M \times M,$$

because $(\pi \times \pi)(s_H(\gamma, e), t_H(\gamma, e)) = (\pi(e), \pi(\gamma \cdot e)) = (s(\gamma), t(\gamma))$. Since (s,t) is proper by assumption and pr_1 is proper, their composition is proper, hence $(\pi \times \pi) \circ (s_H, t_H)$ is proper.

If $K \subseteq OF(M,\mathcal{F}) \times OF(M,\mathcal{F})$ is compact, then $(\pi \times \pi)(K)$ is compact, so $\big((\pi \times \pi) \circ (s_H, t_H)\big)^{-1}((\pi \times \pi)(K))$ is compact. Moreover $(s_H, t_H)^{-1}(K)$ is closed in this compact set (since K is compact and (s_H, t_H) is continuous), hence compact. Thus (s_H, t_H) is proper. □

Proposition 4.34. *The lifted foliation $\widetilde{\mathcal{F}}$ on $OF(M,\mathcal{F})$ is invariant under the G-action.*

Proof. Let $\gamma : x \to y$ be an arrow of G and choose a local bisection $\sigma : U \to G_1$ through γ. Let $\varphi := t \circ \sigma : U \to \varphi(U)$. By definition of the lifted action (4.29), for every $e \in OF(M,\mathcal{F})|_U$ we have

$$\sigma(\pi(e)) \cdot e = (d\varphi)^N(\pi(e)) \circ e = OF(\varphi)(e).$$

By Corollary 4.25, $OF(\varphi)$ preserves the lifted foliation $\widetilde{\mathcal{F}}$. Hence the G-action sends leaves of $\widetilde{\mathcal{F}}$ to leaves of $\widetilde{\mathcal{F}}$, i.e. $\widetilde{\mathcal{F}}$ is G-invariant. □

Lemma 4.35. *For any local foliated isometry φ on M, the induced diffeomorphism*

$$OF(\varphi) : OF(M,\mathcal{F})|_{\mathrm{dom}(\varphi)} \to OF(M,\mathcal{F})|_{\mathrm{im}(\varphi)}, \qquad OF(\varphi)(e) = (d\varphi)^N(\pi(e)) \circ e$$

satisfies $OF(\varphi)^*\theta = \theta$. Equivalently, θ is G-invariant.

Proof. This is the identity $OF(\varphi)^*\theta = \theta$ in Lemma 3.23. □

Proposition 4.36. *The action of G on $(OF(M,\mathcal{F}), \widetilde{\mathcal{F}})$ preserves the transverse Levi-Civita connection ω.*

Proof. Let $\gamma : x \to y$ be an arrow of G and choose a local bisection $\sigma : U \to G_1$ through γ. Let $\varphi := t \circ \sigma : U \to \varphi(U)$. As we have shown, on $OF(M,\mathcal{F})|_U$ the action map is $OF(\varphi)$. By Corollary 4.25, $OF(\varphi)^*\omega = \omega$ on $OF(M,\mathcal{F})|_U$. Since every arrow is contained in the image of such a local bisection, this shows that the G-action preserves ω. □

Proposition 4.37. *Let $\mathcal{O}_G := OF(M,\mathcal{F})/G$ be the orbit space of the lifted G-action, that is, the orbit space of the translation groupoid $H = G \ltimes OF(M,\mathcal{F})$.*

Then $\mathcal{O}_G$ carries a canonical orbifold structure, the right $O(q)$-action on $OF(M,\mathcal{F})$ descends to a right $O(q)$-action on $\mathcal{O}_G$, and the quotient map $\pi_G : \mathcal{O}_G \to Q = M/G$ is a well-defined orbifold map.

Proof. By Lemmas 4.31 and 4.33, $H = G \ltimes OF(M,\mathcal{F})$ is proper étale, hence its orbit space $\mathcal{O}_G = H_0/H$ carries a canonical orbifold structure.

The right $O(q)$-action on $OF(M, \mathcal{F})$ commutes with the left G-action by Lemma 4.30, so it descends to a right $O(q)$-action on $\mathcal{O}_G$. The map $\pi_G : \mathcal{O}_G \to Q$ is induced by the G-equivariant projection $\pi : OF(M, \mathcal{F}) \to M$.

Because H is proper étale, every orbifold chart of $\mathcal{O}_G$ is obtained from a local model

$$H|_V \cong H_e \ltimes V,$$

where H_e is a finite isotropy group. By Proposition 4.34, Lemma 4.35, and Proposition 4.36, the restrictions of $\widetilde{\mathcal{F}}$, θ, and ω to V are invariant under the H_e-action. Hence they descend on the local quotient chart V/H_e to a local orbifold foliation and local orbifold 1-forms. Since these descended objects come from the global G-invariant data on $OF(M, \mathcal{F})$, they are compatible on overlaps. Therefore they glue to an orbifold foliation $\widetilde{\mathcal{F}}_G$ and orbifold 1-forms θ_G, ω_G on $\mathcal{O}_G$. □

Let $(\mathcal{F}_Q, \bar{g})$ be the Riemannian foliation on Q from Proposition 4.27. Let

$$\pi_{\mathcal{O}} : OF(Q, \mathcal{F}_Q) \longrightarrow Q$$

be the orbifold transverse orthonormal frame bundle from Definition 4.9, with lifted foliation $\widetilde{\mathcal{F}}_Q$, transverse canonical form θ_Q, and transverse Levi–Civita connection form ω_Q.

We can now state the proper étale groupoid analogue; the proposition right after proves item (1).

Corollary 4.38. *Let $G \rightrightarrows M$ be a Hausdorff proper effective étale groupoid such that the quotient orbifold $Q := M/G$ is compact and connected, and let $(\mathcal{F}, g)$ be a G-invariant Riemannian foliation of codimension q on M. Let $(\mathcal{F}_Q, \bar{g})$ be the descended Riemannian foliation on Q.*

Let $\mathcal{O}_G := OF(M,\mathcal{F})/G$, $\widetilde{\mathcal{F}}_G$ be the descended foliation on $\mathcal{O}_G$, and $\widetilde{\mathcal{F}}_Q$ on $OF(Q,\mathcal{F}_Q)$.

Then:

(1) $\mathcal{O}_G$ carries a canonical orbifold structure for which $\mathcal{O}_G \to Q$ is an orbifold principal $O(q)$-bundle. Moreover, there is a canonical $O(q)$-equivariant foliated orbifold isomorphism

$$\Phi : (\mathcal{O}_G, \widetilde{\mathcal{F}}_G) \xrightarrow{\cong} (OF(Q,\mathcal{F}_Q), \widetilde{\mathcal{F}}_Q).$$

(2) Via Φ, $(\mathcal{O}_G, \widetilde{\mathcal{F}}_G)$ satisfies the conclusions of Corollary 4.10. Equivalently, there exist orbifold 1-forms θ_G, ω_G on $\mathcal{O}_G$, a manifold $\mathcal{N}$ with a smooth right $O(q)$-action, an $O(q)$-equivariant orbifold fibre bundle $\bar{s}_G : \mathcal{O}_G \to \mathcal{N}$, and a Lie algebra $\mathfrak{g}$ such that $T(\widetilde{\mathcal{F}}_G) = \ker\theta_G \cap \ker\omega_G$, and the fibres of $\bar{s}_G$ are exactly the closures of the leaves of $\widetilde{\mathcal{F}}_G$.

For every $z \in \mathcal{N}$, the restricted foliation $\widetilde{\mathcal{F}}_G\big|_{\bar{s}_G^{-1}(z)}$ is a Lie foliation defined by a canonical $\mathfrak{g}$-valued Maurer–Cartan form with dense holonomy group, where $\mathfrak{g}$ is independent of z up to isomorphism.

Moreover, $(\theta_G, \omega_G, \bar{s}_G, \mathfrak{g})$ are obtained from the corresponding structures on $OF(Q,\mathcal{F}_Q)$ via the canonical isomorphism Φ.

Proposition 4.39. *There is a canonical $O(q)$-equivariant orbifold isomorphism $\Phi : \mathcal{O}_G \xrightarrow{\cong} OF(Q,\mathcal{F}_Q)$. Moreover, $\Phi : (\mathcal{O}_G, \widetilde{\mathcal{F}}_G) \xrightarrow{\cong} (OF(Q,\mathcal{F}_Q), \widetilde{\mathcal{F}}_Q)$ is foliated, and $\Phi^*\theta_Q = \theta_G$, $\Phi^*\omega_Q = \omega_G$.*

Proof. Let $x \in M$ and choose a local model $G|_U \cong G_x \ltimes U$ as in Remark 4.19. Let (U, G_x, ϕ_x) be the corresponding orbifold chart on Q.

Recognizing that the two orbifold constructions are locally the same:

By Proposition 4.27, the quotient chart $U/G_x \xrightarrow{\sim} \phi_x(U)$ carries the descended Riemannian foliation of $(\mathcal{F}|_U, g|_U)$. Hence, by Definition 4.9, the restriction of $OF(Q, \mathcal{F}_Q)$ to $\phi_x(U)$ is represented by the quotient orbifold $OF(U, \mathcal{F}|_U)/G_x$.

On the other hand, under the identification $G|_U \cong G_x \ltimes U$, the lifted $G|_U$-action on $OF(M, \mathcal{F})|_U$ is the natural G_x-action on $OF(U, \mathcal{F}|_U)$. Therefore the restriction of $\mathcal{O}_G = OF(M, \mathcal{F})/G$ over the same base chart $\phi_x(U)$ is represented by the same quotient orbifold: $\pi_G^{-1}(\phi_x(U)) \cong OF(U, \mathcal{F}|_U)/G_x$.

Thus we obtain a canonical local identification

$$\Phi_U : \pi_G^{-1}(\phi_x(U)) \xrightarrow{\cong} OF(Q, \mathcal{F}_Q)|_{\phi_x(U)}$$

induced by the identity on the common local model $OF(U, \mathcal{F}|_U)/G_x$.

If $\lambda : (V, (G_x)_V, \phi_x|_V) \longrightarrow (U', G_y, \phi_y)$ is an embedding of orbifold charts, then by [12, Prop. 2.12(i)] the lifted map

$$OF(\lambda) : OF(V, \mathcal{F}|_V) \to OF(\lambda(V), \mathcal{F}|_{\lambda(V)}), \qquad OF(\lambda)(e) = (d\lambda)^N_{\pi(e)} \circ e,$$

is equivariant with respect to the induced isomorphism

$$(G_x)_V \xrightarrow{\cong} (G_y)_{\lambda(V)}.$$

Hence it descends to the change-of-charts map on both constructions. Therefore the local identifications Φ_U glue to a global orbifold isomorphism

$$\Phi : \mathcal{O}_G \xrightarrow{\cong} OF(Q, \mathcal{F}_Q).$$

Because each local Φ_U identifies two objects over the same orbifold chart $\phi_x(U) \subseteq Q$, the induced map on the base orbifold is the identity.

The right $O(q)$-equivariance is immediate on each local model, hence globally.

Φ *is foliated*:

Finally, by Proposition 4.37 and Definition 4.9, on each common local model $OF(U, \mathcal{F}|_U)/G_x$ the orbifold foliations and orbifold 1-forms $\widetilde{\mathcal{F}}_G$, θ_G, ω_G and $\widetilde{\mathcal{F}}_Q$, θ_Q, ω_Q are the descents of the same G_x-invariant objects on $OF(U, \mathcal{F}|_U)$. Hence they agree under Φ_U, so Φ is foliated and $\Phi^*\theta_Q = \theta_G$, $\Phi^*\omega_Q = \omega_G$. □

At this point, Propositions 4.37 and 4.39 together prove item (1) of Corollary 4.38.

Proof of Corollary 4.38

Proof. Since Q is compact and connected by assumption, Corollary 4.10 applies to the compact connected Riemannian orbifold foliation $(\mathcal{F}_Q, \bar{g})$. Let $\mathcal{O} := OF(Q, \mathcal{F}_Q)$, and let $\widetilde{\mathcal{F}}_Q$ denote the lifted foliation on $\mathcal{O}$.

Step 1: Apply the orbifold version of Molino's theory.

Corollary 4.10 yields orbifold 1-forms θ_Q, ω_Q on $\mathcal{O}$, a manifold $\mathcal{N}$ with a right $O(q)$-action, an $O(q)$-equivariant orbifold fibre bundle $\bar{s} : \mathcal{O} \to \mathcal{N}$, and a Lie algebra $\mathfrak{g}$ such that $T(\widetilde{\mathcal{F}}_Q) = \ker\theta_Q \cap \ker\omega_Q$, the fibres of $\bar{s}$ are the closures of the leaves of $\widetilde{\mathcal{F}}_Q$, and for each $z \in \mathcal{N}$ the restricted foliation $\widetilde{\mathcal{F}}_Q\big|_{\bar{s}^{-1}(z)}$ is a Lie foliation defined by a canonical $\mathfrak{g}$-valued Maurer–Cartan form with dense holonomy group; moreover, $\mathfrak{g}$ is independent of z up to isomorphism.

Step 2: Move that package to $\mathcal{O}_G$ using Φ

By Proposition 4.39, there is a canonical $O(q)$-equivariant foliated orbifold isomorphism

$$\Phi : (\mathcal{O}_G, \widetilde{\mathcal{F}}_G) \xrightarrow{\cong} (\mathcal{O}, \widetilde{\mathcal{F}}_Q)$$

covering id_Q and satisfying $\Phi^*\theta_Q = \theta_G$, $\Phi^*\omega_Q = \omega_G$.

Define $\mathcal{N}_G := \mathcal{N}$, $\bar{s}_G := \bar{s} \circ \Phi : \mathcal{O}_G \to \mathcal{N}_G$. Since $T(\widetilde{\mathcal{F}}_Q) = \ker\theta_Q \cap \ker\omega_Q$ on $\mathcal{O}$, pulling back along Φ gives

$$T(\widetilde{\mathcal{F}}_G) = \ker\theta_G \cap \ker\omega_G$$

on $\mathcal{O}_G$. Thus $(\mathcal{O}_G, \widetilde{\mathcal{F}}_G)$ is transversely parallelizable, with transverse parallelism determined by (θ_G, ω_G).

Step 3: Leaf closure and Lie foliation

Because $\bar{s}$ is an $O(q)$-equivariant orbifold fibre bundle and Φ is an $O(q)$-equivariant orbifold isomorphism, $\bar{s}_G$ is an $O(q)$-equivariant orbifold fibre bundle. For each $z \in \mathcal{N}_G$, $\bar{s}_G^{-1}(z) = \Phi^{-1}\big(\bar{s}^{-1}(z)\big)$. Since Φ is a foliated homeomorphism, it maps leaves to leaves and preserves their closures. Hence the fibres of $\bar{s}_G$ are the closures of the leaves of $\widetilde{\mathcal{F}}_G$.

Let $z \in \mathcal{N}_G$, and let $L_z := \bar{s}_G^{-1}(z)$, $L'_z := \bar{s}^{-1}(z)$. The restriction

$$\Phi_z := \Phi|_{L_z} : L_z \to L'_z$$

is a foliated orbifold diffeomorphism. By Corollary 4.10, the restricted foliation $\widetilde{\mathcal{F}}_Q|_{L'_z}$ is a Lie foliation defined by a canonical non-singular $\mathfrak{g}$-valued Maurer–Cartan form $\omega^z_{\mathrm{MC}} \in \Omega^1(L'_z; \mathfrak{g})$ with $d\omega^z_{MC} + \frac{1}{2}[\omega^z_{MC}, \omega^z_{MC}] = 0$, $T(\widetilde{\mathcal{F}_Q|_{L'_z}}) = \ker(\omega^z_{MC})$.

Now we pull it back:

$$\omega^{z,G}_{\mathrm{MC}} := \Phi_z^*\omega^z_{\mathrm{MC}} \in \Omega^1(L_z; \mathfrak{g}).$$

Then

$$d\omega^{z,G}_{\mathrm{MC}} + \frac{1}{2}[\omega^{z,G}_{\mathrm{MC}}, \omega^{z,G}_{\mathrm{MC}}] = \Phi_z^*\left(d\omega^{z}_{\mathrm{MC}} + \frac{1}{2}[\omega^{z}_{\mathrm{MC}}, \omega^{z}_{\mathrm{MC}}]\right) = 0,$$

and, because Φ_z is foliated, $T\big(\widetilde{\mathcal{F}}_G|_{L_z}\big) = \ker(\omega^{z,G}_{\mathrm{MC}})$. So $\widetilde{\mathcal{F}}_G|_{L_z}$ is a Lie foliation with structural Lie algebra $\mathfrak{g}$.

Step 4: Dense holonomy

To compare holonomy, let $G_{\mathfrak{g}}$ be the connected simply connected Lie group integrating $\mathfrak{g}$. Associated to ω^z_{MC} and $\omega^{z,G}_{MC}$, let η^z and $\eta^{z,G}$ be the associated Darboux connection forms on $L'_z \times G_{\mathfrak{g}}$ and $L_z \times G_{\mathfrak{g}}$, respectively. By functoriality of the Darboux construction,

$$(\Phi_z \times \mathrm{id}_{G_{\mathfrak{g}}})^*\eta^z = \eta^{z,G}.$$

Hence the corresponding holonomy homomorphisms determined by η^z and $\eta^{z,G}$ have the same image in $G_{\mathfrak{g}}$. Since the holonomy group of ω^z_{MC} is dense in $G_{\mathfrak{g}}$, the holonomy group of $\omega^{z,G}_{\mathrm{MC}}$ is also dense in $G_{\mathfrak{g}}$.

Therefore, via Φ, the foliated orbifold $(\mathcal{O}_G, \widetilde{\mathcal{F}}_G)$ satisfies the conclusions of Corollary 4.10, with transverse parallelism determined by (θ_G, ω_G), together with $\bar{s}_G$ and $\mathfrak{g}$. □

4.4 A discussion on the effectiveness assumption

Remark 4.40. We want to explain why "effective" is assumed in the proper étale corollary, but not in the regular groupoid theorem.

First, if $G \rightrightarrows G_0$ is étale, every arrow determines a germ of a local diffeomorphism of the base. There is the effect homomorphism

$$\mathrm{Eff} : G \longrightarrow \Gamma(G_0),$$

where $\Gamma(G_0)$ is the Haefliger groupoid of germs of local diffeomorphisms of G_0. The groupoid G is effective if and only if Eff is injective on arrows, and $\mathrm{Eff}(G) \subseteq \Gamma(G_0)$ is an open effective subgroupoid.

If G is proper and étale, then the orbit space $Q := G_0/G$ carries a canonical orbifold structure, and the associated proper effective orbifold groupoid $\Gamma(Q)$ is weakly equivalent to $\mathrm{Eff}(G)$. Thus effectiveness is the hypothesis we need when we want to identify the given étale groupoid as an effective orbifold presentation of Q.

Second, in the regular Riemannian groupoid theorem we work directly with the full Lie groupoid $G \rightrightarrows M$. In the non-étale setting, an arrow does not canonically determine a germ of a local diffeomorphism of the base. The effect homomorphism is not the right organizing object, so the relevant transverse geometry is encoded instead by local bisections and by the normal/conormal representation.

In particular, a 0-metric is defined by transverse invariance for the canonical action $G \curvearrowright M$, equivalently by the requirement that the normal representation act by fibrewise isometries. Isotropy acting trivially on the transverse directions is therefore already invisible to the metric geometry.

Finally, [11] shows that every regular Lie groupoid fits into an extension

$$I(G)^{\circ} \longrightarrow G \longrightarrow E$$

by a foliation groupoid E, unique up to isomorphism. In the proper regular case, $I(G)^\circ$ is a locally trivial bundle of compact connected Lie groups and E is an orbifold groupoid.

This description involves no effectiveness hypothesis: the connected isotropy bundle is part of the intrinsic regular groupoid data. For this reason, effectiveness is needed in the proper étale/orbifold corollary, but not in the regular Riemannian groupoid theorem itself.

Chapter 5

Examples of Regular Riemannian Lie Groupoids

We'll go over two examples of Molino's structure theorem for regular Riemannian groupoids, as follows.

(1) *Kernel groupoid:* $M \times_B M \rightrightarrows M$.

(2) *Kronecker:* $\mathbb{R} \ltimes \mathbb{T}^2 \rightrightarrows \mathbb{T}^2$.

5.1 The kernel groupoid of a submersion

Our first example, $M \times_B M \rightrightarrows M$, comes from [12, Example 5.1(2)].

5.1.1 The regular groupoid structure

The kernel groupoid and its structure maps

Let $p : M \longrightarrow B$ be a surjective submersion between manifolds. We assume M is compact and connected.

Define

$$G := M \times_B M := \big\{(x, y) \in M \times M : p(x) = p(y)\big\}.$$

Recall: the kernel groupoid $\mathrm{Ker}(p)$ is a Lie subgroupoid of the pair groupoid $\mathrm{Pair}(N)$, defined by

$$\mathrm{Ker}(p)^{(1)} := N \times_M N = \{(y, y') \in N \times N : p(y) = p(y')\},$$

where $p : N \to M$ is a submersion. In our case, $N = M$ and the base manifold is B, so the arrow manifold is $G = M \times_B M$. Then $G = M \times_B M$ is the arrow manifold of a Lie groupoid over M. We say that $(x, y) \in G$ is the arrow $y \to x$.

So the structure maps are $s(x, y) = y$, $t(x, y) = x$, $u(x) = (x, x)$, $i(x, y) = (y, x)$, and for composable arrows (x, y) and (y, z) we set $m((x, y),(y, z)) = (x, z)$, equivalently $(x, y)\cdot(y, z) = (x, z)$.

Let $x, y, z, w \in M$ with $p(x) = p(y) = p(z) = p(w)$. We check units: $s(u(x)) = s(x, x) = x$ and $t(u(x)) = t(x, x) = x$. Inverses: $(x, y) \cdot (y, x) = (x, x) = u(x) = u(t(x, y))$, $(y, x) \cdot (x, y) = (y, y) = u(y) = u(s(x, y))$, and associativity $\big((x, y) \cdot (y, z)\big) \cdot (z, w) = (x, z) \cdot (z, w) = (x, w)$, $(x, y) \cdot \big((y, z) \cdot (z, w)\big) = (x, y) \cdot (y, w) = (x, w)$.

Therefore $G \rightrightarrows M$ is a Lie groupoid. M is a manifold, hence Hausdorff, so $M \times M$ is Hausdorff. The fibre product $G = M \times_B M \subseteq M \times M$ is a smooth submanifold (because $G = (p \times p)^{-1}(\Delta_B)$ with $p \times p$ a submersion), hence G is Hausdorff.

The orbit foliation and regularity

Let $x \in M$. Then $s^{-1}(x) = \big\{(z, x) \in G : p(z) = p(x)\big\}$. Applying t, we get

$$\begin{aligned} t\big(s^{-1}(x)\big) &= \big\{t(z, x) : p(z) = p(x)\big\} \\ &= \big\{z \in M : p(z) = p(x)\big\} \\ &= p^{-1}\big(p(x)\big). \end{aligned}$$

So the orbit set through x is $\mathcal{O}_x = p^{-1}\big(p(x)\big)$. By definition [12, Example 1.1(2)], the leaf through x is the connected component of the fibre $p^{-1}(p(x))$ containing x.

A smooth curve $\gamma(t)$ lies inside a leaf if and only if $p(\gamma(t))$ is constant. Differentiating,

$$\frac{d}{dt}\Big(p(\gamma(t))\Big) = (dp)_{\gamma(t)}\big(\dot{\gamma}(t)\big) = 0.$$

This implies $T(\mathcal{O}) = \ker(dp) \subseteq TM$.

Since p is a submersion, dp has constant rank $\dim(B)$. Therefore $\ker(dp)$ has constant rank $\dim(M) - \dim(B)$, so the orbit distribution has constant rank. Hence $G \rightrightarrows M$ is regular.

5.1.2 The fibre-product 2-metric

Composable arrows and face maps

By definition, $G^{(2)} = \{(h,g) \in G \times G : s(h) = t(g)\}$. Let $h = (x,y)$, $g = (y,z)$. The composability condition is $s(x,y) = y = t(y,z)$. Moreover, $p(x) = p(y)$ and $p(y) = p(z)$, so $p(x) = p(y) = p(z)$. Thus

$$G^{(2)} \cong M \times_B M \times_B M, \qquad (h,g) \longleftrightarrow (x,y,z).$$

Under this identification, the face maps are

$$\pi_1(x,y,z) = (x,y), \quad \pi_2(x,y,z) = (y,z), \quad m(x,y,z) = (x,z),$$

and m corresponds to the groupoid multiplication $(x,y) \cdot (y,z) = (x,z)$.

The fibre-product metric on $G^{(2)}$

We now choose a Riemannian metric g_B on B, and by [4, Lemma 2.1.1], a Riemannian metric g_M on M such that $p : (M, g_M) \longrightarrow (B, g_B)$ is a Riemannian submersion.

By [4, Remark 2.1.5], the fibre product $M \times_B M$ carries a metric $\eta^{(1)} := g_M \times_B g_M$ such that both projections to M are Riemannian submersions. More explicitly,

$$\eta^{(1)} = \mathrm{pr}_1^* g_M + \mathrm{pr}_2^* g_M - (p \circ \mathrm{pr}_1)^* g_B,$$

where the projections are Riemannian submersions.

Now consider

$$M \times_B M \times_B M = \big(M \times_B M\big) \times_B M.$$

Here $\big(M \times_B M\big) \to B$ is given by $p \circ \mathrm{pr}_1$ (equivalently $p \circ \mathrm{pr}_2$ on the fibre product).

We apply the same fibre product construction to the two Riemannian submersions $(M \times_B M, \eta^{(1)}) \to (B, g_B)$ and $(M, g_M) \to (B, g_B)$, and we produce a metric on the fibre product:

$$\eta^{(2)} = \mathrm{pr}_x^* \, g_M + \mathrm{pr}_y^* \, g_M + \mathrm{pr}_z^* \, g_M - 2(p \circ \mathrm{pr}_x)^* g_B.$$

Positive-definiteness of $\eta^{(2)}$

Let $(x, y, z) \in M \times_B M \times_B M$, so $p(x) = p(y) = p(z) =: b$. A tangent vector is $v := (v_x, v_y, v_z) \in T_xM \oplus T_yM \oplus T_zM$ satisfying $dp_x(v_x) = dp_y(v_y) = dp_z(v_z) \in T_bB$. Let us call this common vector

$$a := dp_x(v_x) = dp_y(v_y) = dp_z(v_z) \in T_bB.$$

From [4, Section 2.1], since $p : (M, g_M) \to (B, g_B)$ is a Riemannian submersion, at each point we have the orthogonal splitting $T_xM = \ker(dp_x) \oplus \big(\ker(dp_x)\big)^{\perp}$, and dp_x restricts to an isometry $dp_x : \big(\ker(dp_x)\big)^{\perp} \xrightarrow{\cong} T_bB$. We write

$$v_x = v_x^V + v_x^H \quad \text{with} \quad v_x^V \in \ker(dp_x), \quad v_x^H \in \big(\ker(dp_x)\big)^{\perp},$$

and similarly for v_y and v_z. Then $dp_x(v_x^V) = 0$, $dp_x(v_x^H) = a$, and by the isometry property, $g_M\big(v_x^H, v_x^H\big) = g_B(a, a)$, and similarly for y and z.

We evaluate $\eta^{(2)}$ on $v = (v_x, v_y, v_z)$:

$$\begin{aligned}
\eta^{(2)}(v,v) &= g_M(v_x,v_x) + g_M(v_y,v_y) + g_M(v_z,v_z) - 2g_B(a,a) \\
&= \big(g_M(v_x^V, v_x^V) + g_B(a,a)\big) + \big(g_M(v_y^V, v_y^V) + g_B(a,a)\big) \\
&\qquad + \big(g_M(v_z^V, v_z^V) + g_B(a,a)\big) - 2g_B(a,a) \\
&= g_M(v_x^V, v_x^V) + g_M(v_y^V, v_y^V) \\
&\qquad + g_M(v_z^V, v_z^V) + g_B(a,a).
\end{aligned}$$

If $\eta^{(2)}(v,v) = 0$, then $g_B(a,a) = 0 \Rightarrow a = 0$, and $g_M(v_x^V, v_x^V) = 0 \Rightarrow v_x^V = 0$, and similarly $v_y^V = 0$ and $v_z^V = 0$. Also $a = 0$ implies $dp_x(v_x^H) = 0$. But dp_x is injective on the horizontal space $\big(\ker(dp_x)\big)^{\perp}$, so $v_x^H = 0$, and similarly $v_y^H = 0$ and $v_z^H = 0$.

Therefore $v = 0$. Hence $\eta^{(2)}$ is positive definite, so $\eta^{(2)}$ is a Riemannian metric.

Verification of the 2-metric axioms

It remains to verify the 2-metric axioms from [4, Definition 3.3.2 and Remark 3.3.3].

Clearly the S_3-action permutes (x, y, z), hence permutes (v_x, v_y, v_z). If we look at the formula $\eta^{(2)}(v,v) = g_M(v_x^V, v_x^V) + g_M(v_y^V, v_y^V) + g_M(v_z^V, v_z^V) + g_B(a,a)$, the first three terms are symmetric in x, y, z. The last term uses the common base vector $a = dp_x(v_x) = dp_y(v_y) = dp_z(v_z)$, which is unchanged by permuting the components. So $\eta^{(2)}$ is S_3-invariant.

We show $\eta^{(2)}$ is transverse to the face maps, i.e. $\pi_1, \pi_2, m : G^{(2)} \to G$ are Riemannian submersions (with the metrics induced by $\eta^{(2)}$ on $G^{(2)}$ and $\eta^{(1)}$ on G).

Under the identification $M \times_B M \times_B M = (M \times_B M) \times_B M$, the map $\pi_1(x, y, z) = (x, y)$ is the projection to the first factor $M \times_B M$. By construction, the projections from a fibre product are Riemannian submersions. Hence π_1 is a Riemannian submersion.

Let $\sigma \in S_3$ be the cyclic permutation $\sigma(x, y, z) = (y, z, x)$. Then $\pi_2 = \pi_1 \circ \sigma$. Since σ is an isometry by S_3-invariance, and π_1 is a Riemannian submersion, it follows that π_2 is a Riemannian submersion.

Let $\tau \in S_3$ be the transposition swapping y and z: $\tau(x, y, z) = (x, z, y)$. Then $m = \pi_1 \circ \tau$. Again τ is an isometry by S_3-invariance, so m is a Riemannian submersion.

Therefore $\eta^{(2)}$ is transverse to the face maps π_1, π_2, m.

5.1.3 Molino structures for the kernel groupoid

The normal bundle and orthogonal frame bundle

We already computed $T(\mathcal{O}) = \ker(dp)$, so the normal bundle is $N = TM/\ker(dp)$. Because dp is surjective,

$$T_xM/\ker(dp_x) \cong \mathrm{im}(dp_x) = T_{p(x)}B.$$

Hence $N \cong p^*(TB)$. Therefore $q = \mathrm{rank}(N) = \dim(B)$.

Using $N_x \cong T_{p(x)}B$, an orthogonal frame $e : \mathbb{R}^q \to N_x$ is the same thing as an orthogonal frame of $T_{p(x)}B$. This is to say $OF(M, \mathcal{O}) \cong \big\{(x, e) \bigm| x \in M,\ e : \mathbb{R}^q \to T_{p(x)}B \text{ orthogonal}\big\}$. So

$$OF(M, \mathcal{O}) \cong M \times_B OF(B).$$

The lifted foliation

Recall we construct the lifted foliation $\widetilde{\mathcal{O}}$ on $OF(M,\mathcal{O})$ by choosing a Haefliger cocycle (U_i, s_i, γ_{ij}) for $\mathcal{O}$ and defining submersions $\widetilde{s}_i : OF(M,\mathcal{O})\big|_{U_i} \longrightarrow OF(\mathbb{R}^q)$ whose fibres glue to a global foliation ([12, Section 4.2 and Section 1.2]). Here $\mathcal{O}$ is defined by connected components of fibres of p. So locally we can choose s_i to be local coordinates on B composed with p. Then $\widetilde{s}_i$ records the basepoint in B and the transverse orthogonal frame.

Along a leaf in M, the basepoint $p(x)$ is constant, hence the transverse frame is constant. So,

$$\text{leaf through } (x,e) = (\text{leaf of } \mathcal{O} \text{ through } x) \times \{e\}.$$

Since M is compact, every fibre $p^{-1}(b)$ is compact. Hence each connected component leaf is compact and closed in M. Consequently, each lifted leaf $\big(\text{leaf of } \mathcal{O}\big) \times \{e\}$ is closed in $OF(M,\mathcal{O})$.

The structural Lie algebra

Assume in addition that the fibres of p are connected. Therefore, we can take $N = OF(B)$ and $\kappa(x,e) = e$.

Recall the structural Lie algebra is defined by $\mathfrak{g}_n := l\big(\kappa^{-1}(n),\ \widetilde{\mathcal{O}}\big|_{\kappa^{-1}(n)}\big)$. In our case, for $n = e \in OF(B) = N$ we have

$$\kappa^{-1}(e) = \big(\text{a leaf of } \mathcal{O}\big) \times \{e\}.$$

Moreover, $\widetilde{\mathcal{O}}\big|_{\kappa^{-1}(e)}$ is the codimension 0 foliation with a single leaf, which is $\kappa^{-1}(e)$ itself. Therefore,

$$\mathfrak{g}_n = 0 \qquad \text{for all } n \in N.$$

5.2 The Kronecker flow groupoid on $\mathbb{T}^2$

5.2.1 The regular flow groupoid

The action groupoid and its regular orbit foliation

Now we discuss our second example. Let $M = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ and let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Define an $\mathbb{R}$-action $\Phi_\tau\big([x, y]\big) := [x + \tau, y + \alpha\tau]$, $\tau \in \mathbb{R}$. Let $G := \mathbb{R} \ltimes \mathbb{T}^2 \rightrightarrows \mathbb{T}^2$.

For $(\tau, p) \in \mathbb{R} \times \mathbb{T}^2$, the structure maps are $s(\tau, p) = p$, $t(\tau, p) = \Phi_\tau(p)$, $u(p) = (0, p)$, $i(\tau, p) = (\tau, p)^{-1} = \big(-\tau, \Phi_\tau(p)\big)$, and the multiplication for composable arrows is $(\tau_2, \Phi_{\tau_1}(p)) \cdot (\tau_1, p) = (\tau_1 + \tau_2, p)$. Here composability means $s(\tau_2, \Phi_{\tau_1}(p)) = \Phi_{\tau_1}(p) = t(\tau_1, p)$.

Assume $\Phi_\tau([x, y]) = [x, y]$. Lifting to $\mathbb{R}^2$, the condition $[x + \tau, y + \alpha\tau] = [x, y] \pmod{\mathbb{Z}^2}$ means that there exists $(m, n) \in \mathbb{Z}^2$ such that $x + \tau = x + m$, $y + \alpha\tau = y + n$. Equivalently, $\tau = m \in \mathbb{Z}$, $\alpha\tau = n \in \mathbb{Z}$. Since $\alpha \notin \mathbb{Q}$, the only solution is $\tau = 0$. Hence the isotropy is trivial, and all orbits are 1-dimensional. In particular, the action groupoid is regular (which means constant orbit dimension).

The infinitesimal generator of the flow is the constant vector field $X = \partial_x + \alpha\partial_y$. Thus $\dim(\mathcal{O}) = 1$ and $q := \operatorname{codim}(\mathcal{O}) = 1$. This is the Kronecker foliation on $\mathbb{T}^2$ [12, Example 1.1(3)].

Dense leaves and nonproperness

To see density, the orbit through $[0,0]$ is $\{[\tau, \alpha\tau] : \tau \in \mathbb{R}\} = \{\tau(1,\alpha) \bmod \mathbb{Z}^2\}$. Its closure is a closed subgroup of $\mathbb{T}^2$. Since α satisfies no nontrivial integer relation $m\alpha + n = 0$, the closure cannot be a 1-dimensional subtorus; hence the closure must be all of $\mathbb{T}^2$. Therefore all leaves are dense.

Remark 5.1. The action groupoid $G = \mathbb{R} \ltimes \mathbb{T}^2 \rightrightarrows \mathbb{T}^2$ is not proper when $\alpha \notin \mathbb{Q}$ (equivalently, the $\mathbb{R}$-action is not proper). Properness of the action means the map

$$\mathbb{R} \ltimes \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \times \mathbb{T}^2, \qquad (\tau, p) \mapsto \big(p, \Phi_\tau(p)\big)$$

is proper. For irrational α, we can find $\tau_n \to +\infty$ with $\Phi_{\tau_n}(p) \to p$, contradicting properness.

5.2.2 From a failed Euclidean metric to the gauge 2-metric

Failure of the Euclidean metric on $G^{(2)}$

Suppose we try to define the Euclidean metric on $G^{(2)}$. The space $G^{(2)} = \{(h,g) \in G \times G : s(h) = t(g)\}$ consists of pairs of composable arrows. If $g = (\tau_1, p_1)$ and $h = (\tau_2, p_2)$, then $s(h) = t(g)$ means $p_2 = \tau_1 \cdot p_1 = \Phi_{\tau_1}(p_1)$. Thus a parametrization map is $\mathbb{R}^2 \times \mathbb{T}^2 \longrightarrow G^{(2)}$, $(\tau_1, \tau_2, p) \mapsto \big((\tau_2, \tau_1 \cdot p),\ (\tau_1, p)\big)$. With respect to this parametrization, the face maps are $\pi_1(\tau_1, \tau_2, p) = (\tau_2, \Phi_{\tau_1}(p))$, $\pi_2(\tau_1, \tau_2, p) = (\tau_1, p)$, $m(\tau_1, \tau_2, p) = (\tau_1 + \tau_2, p)$.

Consider the Euclidean metric on $G^{(2)}$

$$g := d\tau_1^2 + d\tau_2^2 + dx^2 + dy^2 \qquad \text{on } \mathbb{R}^2 \times \mathbb{T}^2 \text{ (locally } \mathbb{R}^2 \times \mathbb{R}^2).$$

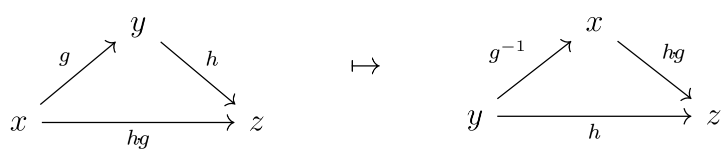
The map π_2 is the projection dropping τ_2, so π_2 is a Riemannian submersion for g. Hence the induced pushforward metric downstairs is $(\pi_2)_* g = d\tau^2 + dx^2 + dy^2$.

For $m(\tau_1, \tau_2, x, y) = (\tau_1 + \tau_2, x, y)$ we have $dm(a, b, c, d) = (a + b, c, d)$. Then the kernel of dm in the (τ_1, τ_2)-directions is spanned by $(1, -1)$, i.e. $\ker(dm) = \mathrm{span}\{\partial_{\tau_1} - \partial_{\tau_2}\}$. A horizontal complement is spanned by $(1, 1)$ together with ∂_x, ∂_y, i.e. $(\ker dm)^\perp = \mathrm{span}\{\partial_{\tau_1} + \partial_{\tau_2},\ \partial_x,\ \partial_y\}$. But $dm(1, 1) = 1 + 1 = 2$. Therefore, to lift a unit ∂_τ downstairs, we need to take $\frac{1}{2}(1, 1)$ upstairs. Its squared norm is $\left\|\frac{1}{2}(1, 1)\right\|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$. Hence $m_* g = \frac{1}{2}\, d\tau^2 + dx^2 + dy^2$. So

$$(\pi_2)_* g \neq m_* g,$$

and this already shows that g cannot be a 2-metric on $G^{(2)}$ ([4, Definition 3.3.2 and Proposition 3.3.4]).

We now consider the triangle of composable arrows:



The permutation $(xy)z$ sends $(h, g) \in G^{(2)}$ to $(h{\cdot}g,\ g^{-1})$ [4, Section 3.3].

In our case, recall $g = (\tau_1, p_1)$ and $h = (\tau_2, p_2)$, and for an action groupoid $(\tau, p)^{-1} = (-\tau,\ \tau \cdot p) = (-\tau, \Phi_\tau(p))$. Thus $g^{-1} = (\tau_1, p_1)^{-1} = (-\tau_1,\ \tau_1 \cdot p_1)$, and $h{\cdot}g = (\tau_2, p_2) \cdot (\tau_1, p_1) = (\tau_2, \tau_1 \cdot p_1) \cdot (\tau_1, p_1) = (\tau_1 + \tau_2, p_1)$. So $(h{\cdot}g,\ g^{-1}) = \big((\tau_1 + \tau_2, p_1),\ (-\tau_1, \tau_1 \cdot p_1)\big) =: (h', g')$.

By definition of the parametrization, (τ_1', τ_2', p') corresponds to $(h', g') = \big((\tau_2', \tau_1' \cdot p'),\ (\tau_1', p')\big)$. We match (τ_1', p') with $(-\tau_1, \tau_1 \cdot p_1)$. Hence $\tau_1' = -\tau_1,\ p' = \tau_1 \cdot p_1 = \Phi_{\tau_1}(p_1)$. Then $(\tau_2', \tau_1' \cdot p') = (\tau_2',\ (-\tau_1) \cdot (\tau_1 \cdot p_1)) = (\tau_2',\ p_1)$. We need this equal to $h' = (\tau_1 + \tau_2, p_1)$, so $\tau_2' = \tau_1 + \tau_2$. Therefore, under $(xy)z$ we have $(\tau_1, \tau_2, p) \mapsto \big(-\tau_1,\ \tau_1 + \tau_2,\ \Phi_{\tau_1}(p)\big)$.

On the (τ_1, τ_2)-coordinates this is

$$(\tau_1, \tau_2) \mapsto (-\tau_1,\ \tau_1 + \tau_2), \qquad \text{matrix} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note it sends $(1, 0)$ to $(-1, 1)$, so it is not orthogonal for the Euclidean metric $d\tau_1^2 + d\tau_2^2$. Thus the Euclidean metric g on $G^{(2)}$ is not S_3-invariant.

The gauge presentation of the Kronecker groupoid

Because the groupoid is not proper, we construct a metric via a free and proper action upstairs and then descending.

Let $P := \mathbb{R} \times \mathbb{T}^2$. Define an $\mathbb{R}$-action on P by

$$a \cdot (k, p) \ := \ (k - a,\ \Phi_a(p)). \tag{5.2}$$

Freeness is obvious: if $a \cdot (k, p) = (k, p)$ then $k - a = k$, hence $a = 0$. For properness, define a diffeomorphism $F : P \longrightarrow \mathbb{R} \times \mathbb{T}^2$, $F(k, p) := (k, \Phi_k(p))$. Its inverse is $F^{-1}(k, x) = (k, \Phi_{-k}(x))$. We compute,

$$\begin{aligned} F\big(a \cdot (k, p)\big) &= F(k - a, \Phi_a(p)) \\ &= (k - a, \Phi_{k-a}(\Phi_a(p))) \\ &= (k - a, \Phi_k(p)). \end{aligned}$$

Thus, under the identification F, the action (5.2) becomes $a \cdot (k, x) = (k - a, x)$ on $\mathbb{R} \times \mathbb{T}^2$, i.e. translation on $\mathbb{R}$ and trivial on $\mathbb{T}^2$, which is a proper action. Hence the action on P is free and proper.

Define

$$\Psi_0 : P/\mathbb{R} \longrightarrow \mathbb{T}^2, \qquad [k,p] \mapsto \Phi_k(p).$$

We want to show Ψ_0 is a diffeomorphism. We need to check a couple things. For well-definedness, if $(k,p) \sim (k-a, \Phi_a(p))$, then $\Phi_{k-a}\big(\Phi_a(p)\big) = \Phi_k(p)$. Surjective: $\Psi_0([0,x]) = x$ for any $x \in \mathbb{T}^2$. To show injectivity, suppose $\Psi_0([k_1,p_1]) = \Psi_0([k_2,p_2])$, i.e. $\Phi_{k_1}(p_1) = \Phi_{k_2}(p_2)$. Applying Φ_{-k_2} to both sides, we get $p_2 = \Phi_{-k_2}\big(\Phi_{k_1}(p_1)\big) = \Phi_{k_1-k_2}(p_1)$. Let $a := k_1 - k_2$. Then $a \cdot (k_1,p_1) = (k_1 - a, \Phi_a(p_1)) = (k_2,p_2)$, so $[k_1,p_1] = [k_2,p_2]$ in $P/\mathbb{R}$.

Let $q : P \to M$, $q(k,p) = p$. Then,

$$P \times_M P = \{((k_1,p),(k_2,p)) :\ k_1,k_2 \in \mathbb{R},\ p \in \mathbb{T}^2\} \cong \mathbb{R}^2 \times \mathbb{T}^2.$$

The diagonal action is $a \cdot (k_1,k_2,p) := (k_1 - a,\ k_2 - a,\ \Phi_a(p))$.

Define

$$\Psi_1 : (P \times_M P)/\mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{T}^2, \qquad [\,(k_1,p),(k_2,p)] \mapsto \big(\tau := k_1 - k_2,\ x := \Phi_{k_2}(p)\big).$$

It is well-defined. τ is invariant under the diagonal action, and $\Phi_{k_2-a}\big(\Phi_a(p)\big) = \Phi_{k_2}(p)$. The source is x and the target is $\Phi_\tau(x)$, so this identifies $(P \times_M P)/\mathbb{R} \rightrightarrows P/\mathbb{R}$ with the action groupoid $\mathbb{R} \times \mathbb{T}^2 \rightrightarrows \mathbb{T}^2$ (after identifying $P/\mathbb{R} \cong \mathbb{T}^2$ via Ψ_0).

The descended 2-metric

Choose the product metric $\eta_P := dk^2 + dx^2 + dy^2 = dk^2 + g_{\mathbb{T}^2}$, where $g_{\mathbb{T}^2}$ is the flat metric on $\mathbb{T}^2$.

In universal cover coordinates $(k, x, y) \in \mathbb{R}^3$ (with (x, y) projecting to $[x, y] \in \mathbb{T}^2$), the action is $a \cdot (k, x, y) = (k - a,\ x + a,\ y + \alpha a)$, a translation in $\mathbb{R}^3$, hence isometric. Therefore η_P is $\mathbb{R}$-invariant and thus transversely invariant.

By [4, Lemma 2.3.1], since the action is free and proper by isometries, the quotient $P/\mathbb{R}$ inherits a unique smooth metric such that the projection $\pi : P \longrightarrow P/\mathbb{R}$ is a Riemannian submersion. Equivalently, the metric on $P/\mathbb{R}$ is defined so that $d\pi$ restricts to an isometry on the orthogonal complement of the orbits.

Define a metric $\eta^{(2)}$ by descending a product metric from $P \times_M P \times_M P$. This is the gauge-groupoid construction of 2-metrics in [4, Prop. 4.2.4].

Take $P \times_M P \times_M P \cong \mathbb{R}^3 \times \mathbb{T}^2$ with diagonal action $a \cdot (k_1, k_2, k_3, p) = (k_1 - a,\ k_2 - a,\ k_3 - a,\ \Phi_a(p))$. Let

$$ds^2 := dk_1^2 + dk_2^2 + dk_3^2 + dx^2 + dy^2.$$

By [4, Lemma 2.3.1 and Proposition 2.3.3], the metric descends to

$$(P \times_M P \times_M P)/\mathbb{R} = G^{(2)},$$

and the face maps $\{\pi_1, \pi_2, m\}$ descend to Riemannian submersions on the quotient.

Quotient metric in invariant coordinates

On the universal cover $(k_1, k_2, k_3, x, y) \in \mathbb{R}^5$, the diagonal action is $(k_1, k_2, k_3, x, y) \mapsto (k_1 - a,\ k_2 - a,\ k_3 - a,\ x + a,\ y + \alpha a)$. Let $u_1 = k_1 - k_2$, $u_2 = k_2 - k_3$, $\tilde{x} = x + k_3$, $\tilde{y} = y + \alpha k_3$, $v = k_3$. This makes $(u_1, u_2, \tilde{x}, \tilde{y})$ invariant under the action, and $v \mapsto v - a$. From $k_3 = v$, $k_2 = u_2 + v$, $k_1 = u_1 + u_2 + v$, $x = \tilde{x} - v$, $y = \tilde{y} - \alpha v$, we get $dk_3 = dv$, $dk_2 = du_2 + dv$, $dk_1 = du_1 + du_2 + dv$, $dx = d\tilde{x} - dv$, $dy = d\tilde{y} - \alpha dv$.

Then

$$
\begin{aligned}
dk_1^2 &= (du_1 + du_2 + dv)^2 = du_1^2 + du_2^2 + dv^2 + 2\,du_1\,du_2 + 2\,du_1\,dv + 2\,du_2\,dv,\\
dk_2^2 &= (du_2 + dv)^2 = du_2^2 + 2\,du_2\,dv + dv^2,\\
dk_3^2 &= dv^2,\\
dx^2 &= (d\tilde{x} - dv)^2 = d\tilde{x}^{\,2} - 2\,d\tilde{x}\,dv + dv^2,\\
dy^2 &= (d\tilde{y} - \alpha dv)^2 = d\tilde{y}^{\,2} - 2\alpha\,d\tilde{y}\,dv + \alpha^2\,dv^2.
\end{aligned}
$$

Summing together:

$$
\begin{aligned}
g = ds^2 = (du_1^2 + 2du_2^2 + 2\,du_1\,du_2 + d\tilde{x}^{\,2} + d\tilde{y}^{\,2})\\
+ (2\,du_1\,dv + 4\,du_2\,dv - 2\,d\tilde{x}\,dv - 2\alpha\,d\tilde{y}\,dv) + (\alpha^2 + 4)\,dv^2.
\end{aligned}
$$

In coordinate order $(u_1, u_2, \tilde{x}, \tilde{y}, v)$,

$$
g_{vv} = \alpha^2 + 4, \qquad b = \begin{pmatrix} g_{u_1 v} \\ g_{u_2 v} \\ g_{\tilde{x} v} \\ g_{\tilde{y} v} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -\alpha \end{pmatrix}, \qquad g_{\text{base}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

(The last component of b is $-\alpha$ because the cross term is $-2\alpha\,d\tilde{y}\,dv$, and g_{base} is symmetric with $g_{u_1u_2} = g_{u_2u_1} = 1$.)

Let the quotient coordinates be $q = (u_1, u_2, \tilde{x}, \tilde{y})$, and let $X = a_1\partial_{u_1} + a_2\partial_{u_2} + a_3\partial_{\tilde{x}} + a_4\partial_{\tilde{y}}$ be a tangent vector on the quotient. Let $X + \lambda\partial_v$ be a lift upstairs. We impose orthogonality to

∂_v so that the lift is horizontal:

$$
\begin{aligned}
0 &= g(X + \lambda\partial_v, \partial_v) \\
&= g(X, \partial_v) + \lambda\, g(\partial_v, \partial_v) \\
&= \sum_{i=1}^{4} a_i\, g_{iv} + \lambda\, g_{vv} \\
&= b^T a + \lambda(\alpha^2 + 4),
\end{aligned}
$$

where $a = (a_1, a_2, a_3, a_4)^T$. Thus

$$
\lambda = -\frac{b^T a}{\alpha^2 + 4}.
$$

Then

$$
\begin{aligned}
g(X + \lambda\partial_v,\ X + \lambda\partial_v) &= g(X, X) + 2\lambda\, g(X, \partial_v) + \lambda^2\, g(\partial_v, \partial_v) \\
&= a^T g_{\text{base}} a + 2\lambda(b^T a) + \lambda^2(\alpha^2 + 4) \\
&= a^T g_{\text{base}} a - \frac{(b^T a)^2}{\alpha^2 + 4} \\
&= a^T \left(g_{\text{base}} - \frac{1}{\alpha^2 + 4}\, bb^T \right) a.
\end{aligned}
$$

Hence the quotient metric is

$$
\eta^{(2)} = g_{\text{base}} - \frac{1}{\alpha^2 + 4}\, bb^T = \frac{1}{\alpha^2 + 4} \begin{pmatrix} \alpha^2 + 3 & \alpha^2 + 2 & 1 & \alpha \\ \alpha^2 + 2 & 2\alpha^2 + 4 & 2 & 2\alpha \\ 1 & 2 & \alpha^2 + 3 & -\alpha \\ \alpha & 2\alpha & -\alpha & 4 \end{pmatrix}.
$$

5.2.3 Molino structures for the Kronecker foliation

The normal bundle and orthogonal frame bundle

Recall: choose a flat metric on $\mathbb{T}^2$. The tangent line field of the Kronecker foliation is spanned by

$$X = (1, \alpha), \qquad \text{and a unit normal is} \qquad Y = \frac{1}{\sqrt{1+\alpha^2}}(-\alpha, 1),$$

which is constant.

Hence the normal bundle is trivial: $N \cong \mathbb{T}^2 \times \mathbb{R}$, i.e. the normal bundle is a line bundle of rank 1. For a line, orthogonal frames are just ± 1, so $O(1) = \{\pm 1\}$. Thus the orthogonal frame bundle is

$$OF(M, \mathcal{O}) \cong \mathbb{T}^2 \times O(1) = (\mathbb{T}^2 \times \{+1\}) \sqcup (\mathbb{T}^2 \times \{-1\}).$$

The normal representation

We discuss the normal representation. Recall that $\Phi_\tau([x, y]) = [x+\tau, y+\alpha\tau]$, so $d\Phi_\tau((\partial_x)_p) = (\partial_x)_{\Phi_\tau(p)}$, $d\Phi_\tau((\partial_y)_p) = (\partial_y)_{\Phi_\tau(p)}$. Since $X = \partial_x + \alpha\partial_y$ is constant, we have $d\Phi_\tau(X_p) = X_{\Phi_\tau(p)}$. Thus $d\Phi_\tau$ preserves $T\mathcal{O}$, and it induces $\lambda^N_{(\tau,p)} : N_p \to N_{\Phi_\tau(p)}$.

Let $Y = \frac{1}{\sqrt{1+\alpha^2}}(-\alpha\partial_x + \partial_y)$. Then Y is a global unit vector field orthogonal to $T\mathcal{O}$, and its class $[Y]$ gives a global unit section of $N = T\mathbb{T}^2/T\mathcal{O}$; hence N is trivialized by $p \mapsto [Y_p]$. Because Y is also constant, $d\Phi_\tau(Y_p) = Y_{\Phi_\tau(p)}$, so $\lambda^N_{(\tau,p)}([Y_p]) = [Y_{\Phi_\tau(p)}]$. Under the trivialization of N by $[Y]$, this says $\lambda^N_{(\tau,p)} = \mathrm{id}_{\mathbb{R}}$. So the normal representation is trivial.

A point $(p, \varepsilon) \in \mathbb{T}^2 \times \{\pm 1\}$ represents the frame $1 \mapsto \varepsilon[Y_p]$. By Proposition 3.14, the groupoid action on $OF(M, \mathcal{O})$ is induced by the normal representation. Hence, $(\tau, p)\cdot(p, \varepsilon) = (\Phi_\tau(p), \varepsilon)$, $\varepsilon \in \{\pm 1\}$. So $\mathbb{T}^2 \times \{+1\}$ and $\mathbb{T}^2 \times \{-1\}$ are G-invariant.

The lifted foliation and structural Lie algebra

Because $O(1)$ is discrete, the lifted foliation is just two copies of the original foliation:

$$\widetilde{\mathcal{O}} = (\mathcal{O} \times \{+1\}) \sqcup (\mathcal{O} \times \{-1\}).$$

Because $\mathcal{O}$ has dense leaves in $\mathbb{T}^2$ (for $\alpha \notin \mathbb{Q}$), each lifted leaf closure is a full connected component: $\overline{\widetilde{\mathcal{O}}_{x,+1}} = \mathbb{T}^2 \times \{+1\}$, $\overline{\widetilde{\mathcal{O}}_{x,-1}} = \mathbb{T}^2 \times \{-1\}$. So the basic space of leaf closures has two points $W = \{w_+, w_-\}$. Writing κ for the projection to the basic space, we have

$$\kappa(p, +1) = w_+, \qquad \kappa(p, -1) = w_-,$$

and the right $O(1)$-action flips the sign and swaps $w_+ \leftrightarrow w_-$.

On each fibre $\kappa^{-1}(w_\pm) = \mathbb{T}^2 \times \{\pm 1\}$, the restricted foliation is the Kronecker foliation, hence a Lie foliation with Lie algebra $\mathbb{R}$.

Consider the constant 1-form $\omega := \alpha\, dx - dy$. Then $\omega(X) = \alpha \cdot 1 - 1 \cdot \alpha = 0$, so $\ker(\omega) = T\mathcal{O}$. The form ω is closed and nowhere zero ([12, Example 4.22]). Here the Lie algebra is $\mathbb{R}$ (abelian).

Take generators of $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$ given by loops $\gamma_1(t) = (t, 0)$, $\gamma_2(t) = (0, t)$, $t \in [0, 1]$. Then

$$\int_{\gamma_1} \omega = \int_0^1 \alpha\, dt = \alpha, \qquad \int_{\gamma_2} \omega = \int_0^1 (-1)\, dt = -1.$$

Hence the period group is

$$\Gamma_\omega = \{\, m\alpha - n : (m,n) \in \mathbb{Z}^2 \,\} \subseteq \mathbb{R}.$$

If $\alpha \notin \mathbb{Q}$, this subgroup is dense in $\mathbb{R}$ (Diophantine approximation: for every $\varepsilon > 0$ there exist $m, n \in \mathbb{Z}$ with $|m\alpha - n| < \varepsilon$). By [12, Lemma 4.23], dense holonomy is equivalent to dense leaves.

The structural Lie algebra along each component is

$$\mathfrak{g}_{w_+} \cong \mathbb{R}, \qquad \mathfrak{g}_{w_-} \cong \mathbb{R}.$$

5.2.4 The Lie algebroid of the action groupoid and the basic Lie algebroid

We apply the Lie algebroid result from Chapter 3 to G. From Section 3.4.1 and Proposition 3.51, the Lie algebroid of the Lie groupoid $G = \mathbb{R} \ltimes \mathbb{T}^2 \rightrightarrows \mathbb{T}^2$ is $(A_G)_p = \ker(ds)_{1_p}$. Here the arrow manifold is $G_1 = \mathbb{R} \times \mathbb{T}^2$ and $1_p = (0,p)$. Thus,

$$T_{(0,p)}G_1 = T_0\mathbb{R} \oplus T_p\mathbb{T}^2 \cong \mathbb{R} \oplus T_p\mathbb{T}^2.$$

Let $(a,v) \in \mathbb{R} \oplus T_p\mathbb{T}^2$. Since $s(\tau,p) = p$, we have $(ds)_{(0,p)}(a,v) = v$. Thus, $(A_G)_p = \ker(ds)_{(0,p)} = \{(a,0) : a \in \mathbb{R}\} \cong \mathbb{R}$. So $A_G \cong \mathbb{T}^2 \times \mathbb{R}$. Let $e_p := (1,0) \in (A_G)_p$. Then every element of $(A_G)_p$ is of the form ae_p, $a \in \mathbb{R}$. Therefore every smooth section is of the form fe, $f \in C^\infty(\mathbb{T}^2)$.

The anchor is $\rho_G : A_G \to T\mathbb{T}^2$, $\rho_G = dt|_{A_G}$. For $ae_p = (a,0) \in (A_G)_p$, we compute

$$\rho_G(ae_p) = dt_{(0,p)}(a,0) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} t(\varepsilon a, p) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Phi_{\varepsilon a}(p),$$

where the last equality follows from $t(\tau,p) = \Phi_\tau(p)$. Since $X = \partial_x + \alpha\partial_y$, we get $\rho_G(ae_p) = aX_p$. So for a section $fe \in \Gamma(A_G)$, $\rho_G(fe) = fX$. Hence $\mathrm{im}(\rho_G) = \mathbb{R}X = T\mathcal{O}$. If $ae_p \in (\ker\rho_G)_p$, then $aX_p = 0$. Since $X_p \neq 0$, this forces $a = 0$. Hence $(\ker\rho_G)_p = 0$ for every $p \in \mathbb{T}^2$, and therefore $\ker\rho_G = 0$. We already showed that the $\mathbb{R}$-action on $\mathbb{T}^2$ is free, so $G_p = \{1_p\}$. Thus, $\mathrm{Lie}(G_p) = 0 = (\ker\rho_G)_p$; this is the identity from Proposition 3.55.

Now the exact sequence from Proposition 3.55 becomes

$$0 \longrightarrow 0 \longrightarrow \mathbb{T}^2 \times \mathbb{R} \xrightarrow{\rho_G} \mathbb{R}X \longrightarrow 0.$$

Thus, $\rho_G : A_G \to T\mathcal{O}$ is a vector bundle isomorphism. By the Lie algebroid construction in Section 3.4.2, the anchor is a Lie algebra homomorphism on sections: $\rho_G([a,b]) = [\rho_G(a), \rho_G(b)]$. Therefore, $A_G \cong T\mathcal{O}$ as Lie algebroids.

Finally, we pass to the basic Lie algebroid. By Proposition 3.37, for the lifted foliation $\big(OF(M,\mathcal{O}), \widetilde{\mathcal{O}}\big)$, the basic Lie algebroid is $A = b\big(OF(M,\mathcal{O}), \widetilde{\mathcal{O}}\big) \longrightarrow W$. Here $W = \{w_+, w_-\}$, where $L_{w_+} = \mathbb{T}^2 \times \{+1\}$ and $L_{w_-} = \mathbb{T}^2 \times \{-1\}$. Since W is discrete, $TW = 0$. Therefore the anchor of the basic Lie algebroid is $\rho_A : A \to TW = 0$. So, $\rho_A = 0$, $\ker\rho_A = A$.

By Lemma 3.38, for each $w \in W$, $(\ker\rho_A)_w \cong l\big(L_w, \widetilde{\mathcal{O}}|_{L_w}\big)$. But here $\ker\rho_A = A$, so $A_w = (\ker\rho_A)_w \cong l\big(L_w, \widetilde{\mathcal{O}}|_{L_w}\big)$. On each fibre, $L_{w_\pm} = \mathbb{T}^2 \times \{\pm 1\}$, the restricted foliation is again the Kronecker foliation. As discussed above, it is a Lie foliation with structural Lie algebra $\mathbb{R}$. So, $A_{w_+} \cong \mathbb{R}$, $A_{w_-} \cong \mathbb{R}$. Since W only has two points, this gives a trivialization $A \cong W \times \mathbb{R}$.

On sections, $\Gamma(A) \cong \mathbb{R} \oplus \mathbb{R}$. A section is a pair (a_+, a_-), where $a_\pm \in \mathbb{R}$. Since the structural Lie algebra on each component is $\mathbb{R}$, which is abelian, the bracket is $[(a_+, a_-), (b_+, b_-)] = (0, 0)$. Therefore, for the original groupoid Lie algebroid we have $\ker \rho_G = 0$, but for the basic Lie algebroid we have $\ker \rho_A = A \cong W \times \mathbb{R}$. So we have trivial infinitesimal isotropy at the level of the original groupoid, but nontrivial infinitesimal isotropy for the basic Lie algebroid.

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