ADMISSIBLE A_P WEIGHTS ON HARMONIC NA GROUPS

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Abstract. This note gives a discussion of the main results in [1]. We will not give a complete proof of the theorem; instead, we collect the necessary background and explain the main ideas leading to the weighted weak and strong type estimates for admissible A_p weights on harmonic NA groups.

1. Motivations and background

In $\mathbb{R}^d$, the Hardy–Littlewood maximal operator is controlled on weighted L^p spaces by the Muckenhoupt A_p condition.

Natural question: Can one build an analogous weighted maximal theory on a harmonic NA group?

Definition 1.1. [1, Section 2]. A harmonic NA group is a semidirect product

$$S = N \rtimes A.$$

Here N is an H-type group and $A = (0, \infty)$. The group A acts on N by non-isotropic dilations

$$\delta_a(X, Z) = (\sqrt{a}\, X, aZ), \qquad a \in A.$$

The group is equipped with a left-invariant Riemannian metric and left Haar measure.

Proposition 1.2. [1, (2.2)]. *Let $B(x, r)$ be the open ball centered at $x \in S$, with radius $r > 0$. The volume of $B(x, r)$ in S satisfies*

$$|B(x, r)| \simeq \begin{cases} r^{\ell}, & 0 < r < 1, \\ e^{2\rho r}, & r \geq 1. \end{cases}$$

Thus small balls are polynomial, but large balls are exponential.

Definition 1.3. [1, Section 3]. For $r > 0$, define the ball averaging operators

$$A_r f(x) = \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)|\, dy,$$

and the centered maximal operator

$$Mf(x) = \sup_{r>0} A_r f(x).$$

Then split M into local and large-scale variants by

$$M^0 f(x) = \sup_{0<r\leq T} A_r f(x), \qquad M^{\infty} f(x) = \sup_{r>T} A_r f(x), \qquad T > 0 \text{ finite}.$$

The local part is handled by local A_p theory.

Proposition 1.4. [1, Proposition 2.4]. *For $x, y \in S$ and $s, t \geq 1$,*

$$|B(x, s) \cap B(y, t)| \simeq e^{\rho(s+t-d(x,y))}.$$

The size of the intersection of two balls depends exponentially on the radii and on the distance between centers.

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Proposition 1.5. [1, (3.3)]. *Define annuli by*

$$\Omega_j = B(e,j) \setminus B(e,j-1).$$

Then, for $x \in S$ and $j, N \in \mathbb{N}$,

$$|\Omega_j \cap B(x,N)| \simeq e^{\rho(N+j-d(e,x))}.$$

This is to say: if x is far from the identity, then only a controlled chunk of the annulus Ω_j fits inside a large ball centered at x, and its size is given by this exponential formula.

2. Weights and the bridge identity

We introduce a necessary condition that motivates the definition.

Proposition 2.1. [1, (3.1)]. *If M were bounded on $L^p(w)$, then it has been shown that, for all $N \in \mathbb{N}$ and measurable $E, F \subseteq S$,*

$$\mathbb{1} \otimes w\big(\{(x,y) \in E \times F : d(x,y) < N\}\big) \lesssim e^{2\rho N} w(E)^{1/p} w(F)^{1-1/p}.$$

Here $\mathbb{1} \otimes w$ is the product measure on $S \times S$; this measures weighted interactions between E and F at scale N. This is a necessary condition but not sufficient.

Example 2.2. [1, Example 3.3]. The weight

$$w(x) = e^{-2\rho d(e,x)}$$

satisfies the necessary condition, yet M is not even weighted weak type (p,p). Hence the necessary condition is too weak to guarantee boundedness.

So we need to strengthen it.

Definition 2.3. [1, Definition 1.1]. Let $1 < p < \infty$, $\beta \in (0,1)$, and $\beta \le \alpha < p$. A weight w is an admissible A_p weight with parameters (α, β) if

(1) $w \in A_{p,\mathrm{loc}}(S)$, meaning the Euclidean A_p condition holds on balls of radius at most 2.
(2) For every $N \in \mathbb{N}$,

$$\mathbb{1} \otimes w\big(\{(x,y) \in E \times F : d(x,y) < N\}\big) \lesssim e^{2\beta\rho N} w(E)^{\alpha/p} w(F)^{1-\alpha/p}, \qquad \forall E, F \subseteq S.$$

Remark 2.4. Condition (1) handles M^0. Haar measure is locally doubling; the small-scale part behaves like usual local Euclidean theory, so a local A_p-type condition is enough there.

Condition (2) handles M^∞. The parameter β measures how much exponential growth is allowed. Smaller β means a stronger large-scale condition, and $\beta < 1$. The parameter α tells how the estimate is split between E and F. The quantity $\alpha - \beta$ is what later produces the strong type; exponential decay is needed.

Now we give an identity that bridges the geometry and the operator; this will be used later.

Lemma 2.5. [1, (3.2)]. *For measurable sets $E, F \subseteq S$ and $N \in \mathbb{N}$,*

$$\begin{aligned}\int_F A_N(\chi_E)(x) w(x)\,dx &= \frac{1}{|B(e,N)|} \int_F \int_E \chi_{\{d(x,y)<N\}}\,dy\,w(x)\,dx \\ &= \frac{1}{|B(e,N)|}\,(\mathbb{1} \otimes w)\big(\{(y,x) \in E \times F : d(y,x) < N\}\big).\end{aligned}$$

Proof sketch. This is Fubini's theorem together with left-invariance of the metric and measure, so $|B(x,N)| = |B(e,N)|$. On the left-hand side,

$$A_N(\chi_E)(x) = \frac{|E \cap B(x,N)|}{|B(e,N)|},$$

which is the proportion of $B(x,N)$ that lies inside E. Thus the identity measures how strongly points of F see E at scale N, with weight. □

3. The main theorem and the key lemma

Theorem 3.1. [1, Theorem 1.2]. *Let $1 < p < \infty$, $\beta \in (0,1)$, $\beta \le \alpha < p$, and let w be an admissible A_p weight with parameters (α, β). Then the following statements are true.*

(1) *If $\beta < \alpha$, then*
$$\|Mf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$
(2) *If $\beta = \alpha$, then*
$$\|Mf\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^p(w)}.$$

Thus the case $\beta = \alpha$ is the endpoint weak-type estimate, and the case $\beta < \alpha$ is the strong-type estimate.

First, we need the following key averaging lemma.

Lemma 3.2. [1, Lemma 3.4, (3.15)]. *Let $1 < p < \infty$, $\beta \in (0,1)$, $\beta \le \alpha < p$, and let w be an admissible A_p weight with parameters (α, β). For each $\lambda > 0$ and $N \in \mathbb{N}$, there exists $\eta > 0$, independent of λ and N, such that*
$$w(\{A_N(A_1 f) \ge \lambda\}) \lesssim \sum_{k=0}^{N} e^{\rho(k-N)(1-\beta)p/\alpha} e^{2\rho\beta pk/\alpha} w(\{A_2 f > \eta\lambda e^{2\rho k}\}).$$

Proof sketch. Because the averaging operators are homogeneous,
$$A_N(A_1(f/\lambda)) = \lambda^{-1} A_N(A_1 f).$$
So it is sufficient to consider $\lambda = 1$. We will restore general λ by scaling.

To deal with exponential volume growth, decompose
$$A_1 f = (A_1 f)\chi_{E_{-1}} + \sum_{k=0}^{N} (A_1 f)\chi_{E_k} + (A_1 f)\chi_{E_\infty}.$$
Define level sets
$$E_k = \{x \in S : e^{2\rho(k-1)} \le A_1 f(x) < e^{2\rho k}\}, \qquad 0 \le k \le N,$$
which are the medium levels,
$$E_{-1} = \{x \in S : A_1 f(x) \le e^{-2\rho}\},$$
which is the very small part, and
$$E_\infty = \{x \in S : A_1 f(x) \ge e^{2\rho N}\}.$$
Then
$$A_1 f \le e^{-1} + \sum_{k=0}^{N} e^{2\rho k} \chi_{E_k} + (A_1 f)\chi_{E_\infty}.$$
For all $N \in \mathbb{N}$,
$$A_N(A_1 f) \le e^{-1} + \sum_{k=0}^{N} e^{2\rho k} A_N(\chi_{E_k}) + A_N((A_1 f)\chi_{E_\infty}).$$
So
$$w(\{A_N(A_1 f) \ge 1\}) \le \underbrace{w\left(\left\{\sum_{k=0}^{N} e^{2\rho k} A_N(\chi_{E_k}) \ge e^{-1}\right\}\right)}_{\text{finite sum part}} + \underbrace{w(\{A_N((A_1 f)\chi_{E_\infty}) \ge 1 - 2e^{-1}\})}_{\text{tail term}}.$$

Tail term. Suppose $A_N((A_1 f)\chi_{E_\infty})(x) \neq 0$. The average over $B(x,N)$ of $(A_1 f)\chi_{E_\infty}$ is nonzero, so $B(x,N) \cap E_\infty \neq \varnothing$. Choose

$$y \in B(x,N) \cap E_\infty.$$

Then $A_1 f(y) \geq e^{2\rho N}$. Let $z \in B(y,1)$. Since $B(y,1) \subset B(z,2)$,

$$\begin{aligned} A_2 f(z) &= \frac{1}{|B(z,2)|} \int_{B(z,2)} f(u)\, du \\ &\geq \frac{1}{|B(z,2)|} \int_{B(y,1)} f(u)\, du = \frac{|B(y,1)|}{|B(z,2)|} A_1 f(y). \end{aligned}$$

Because the radii of the balls are fixed, that volume ratio is a fixed positive constant. Thus $A_2 f(z) \geq c e^{2\rho N}$.

So every $z \in B(y,1)$ lies in $\{A_2 f > c e^{2\rho N}\}$. Also, because $y \in B(x,N)$,

$$B(y,1) \subset B(x,N+1) \cap \{A_2 f > c e^{2\rho N}\}.$$

We want to convert this into a lower bound for an averaging operator. Let

$$G := \{A_2 f > c_0 e^{2\rho N}\}.$$

Then

$$\begin{aligned} A_{N+1}(\chi_G)(x) &= \frac{1}{|B(x,N+1)|} \int_{B(x,N+1)} \chi_G(u)\, du \\ &\geq \frac{|B(y,1)|}{|B(x,N+1)|} \simeq e^{-2\rho(N+1)} \gtrsim e^{-2\rho N}. \end{aligned}$$

Hence

$$\{x : B(x,N) \cap E_\infty \neq \varnothing\} \subset \{x : A_{N+1}(\chi_G)(x) \gtrsim e^{-2\rho N}\}.$$

Use the trivial inequality

$$w(H_N) \lesssim e^{2\rho N} \int_{H_N} A_{N+1}(\chi_G)(x) w(x)\, dx,$$

where

$$H_N := \{x : A_{N+1}(\chi_G)(x) > c e^{-2\rho N}\}.$$

Using the bridge identity and the admissible condition,

$$\int_{H_N} A_{N+1}(\chi_G)(x) w(x)\, dx \lesssim e^{2\rho(N+1)(\beta-1)} w(G)^{\alpha/p} w(H_N)^{1-\alpha/p}.$$

Solving for $w(H_N)$ gives

$$w(H_N) \lesssim e^{2\rho N \beta p/\alpha} w(G).$$

Finite sum part. Estimate the finite sum over $E_0, \ldots, E_N$. Look at

$$\sum_{k=0}^{N} e^{2\rho k} A_N(\chi_{E_k})(x).$$

The idea is to use the pigeonhole argument. Choose $\gamma \in (0,1)$. If every term satisfies

$$A_N(\chi_{E_k})(x) < \frac{e^{2\rho\gamma} - 1}{e^{2\rho(k+2)}} e^{2\rho(k-N)\gamma} =: T_k,$$

then summing gives something $< e^{-1}$. So whenever the finite sum is $\geq e^{-1}$, there exists k such that

$$A_N(\chi_{E_k})(x) \geq T_k.$$

Define

$$F_k := \{x : A_N(\chi_{E_k})(x) \geq T_k\}.$$

Then

$$w\Big(\Big\{\sum_{k=0}^{N} e^{2\rho k} A_N(\chi_{E_k}) \geq e^{-1}\Big\}\Big) \leq \sum_{k=0}^{N} w(F_k).$$

By the bridge identity,

$$\int_{F_k} A_N(\chi_{E_k})(x) w(x)\, dx = \frac{1}{|B(e,N)|}(\mathbb{1} \otimes w)\big(\{(y,x) \in E_k \times F_k : d(y,x) < N\}\big).$$

Using admissibility and the definition of F_k,

$$w(F_k) \lesssim e^{2\rho(k+2)p/\alpha} e^{-2\rho(k-N)\gamma p/\alpha} e^{2\rho N(\beta-1)p/\alpha} w(E_k).$$

Let

$$\gamma = \frac{1-\beta}{2}.$$

Then

$$w(F_k) \lesssim e^{\rho(k-N)(1-\beta)p/\alpha} e^{2\rho\beta pk/\alpha} w(E_k).$$

Since E_k is a level set for $A_1 f$, and $A_1 f \lesssim A_2 f$, we have

$$E_k \subset \{A_2 f > \eta e^{2\rho k}\}.$$

Plugging this back gives the desired bound. Scaling restores the general λ. □

4. Sketch of the proof of the main theorem

Following [1, Proof of Theorem 1.2], the local part M^0 is already controlled by $w \in A_{p,\mathrm{loc}}(S)$. Therefore, it remains to show estimates for M^∞:

$$w(\{M^\infty f \geq \lambda\}) \lesssim \lambda^{-p} \int_S |f(x)|^p w(x)\, dx, \qquad \|M^\infty f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)},$$

for the cases $\beta = \alpha$ and $\beta < \alpha$, respectively.

Case 1: $\beta = \alpha$.

Let $\lambda > 0$. By Lemma 3.2 (that is, [1, Lemma 3.4]),

$$w(\{M^\infty f > \lambda\}) \lesssim \sum_{N=0}^{\infty} \sum_{k=0}^{N} e^{\rho(k-N)(1-\beta)p/\alpha} e^{2\rho\beta pk/\alpha} w(\{A_2 f > \eta\lambda e^{2\rho k}\}).$$

The geometric series

$$\sum_{N \geq k} e^{\rho(k-N)(1-\beta)p/\alpha}$$

is bounded uniformly in k.

On $\{A_2 f > \eta\lambda e^{2\rho k}\}$,

$$1 \leq C\lambda^{-p} e^{-2\rho pk} |A_2 f|^p.$$

The remaining exponent is

$$e^{2\rho pk(\beta/\alpha - 1)}.$$

When $\beta = \alpha$, this exponent is 1. Hence

$$w(\{M^\infty f > \lambda\}) \lesssim \frac{1}{\lambda^p} \int_S |A_2 f(x)|^p w(x)\, dx.$$

Since $A_2 f \leq M^0 f$ and M^0 is controlled by $w \in A_{p,\mathrm{loc}}(S)$,

$$w(\{Mf > \lambda\}) \lesssim \frac{1}{\lambda^p} \|f\|^p_{L^p(w)}.$$

This gives weak type (p,p).

Case 2: $\beta < \alpha$.

For $\beta < \alpha$, the quantitative estimate in [1, (3.22)] gives

$$\|A_N f\|_{L^p(w)}^p \lesssim \sum_{k=0}^{N} e^{\rho(k-N)(1-\beta)p/\alpha} e^{2\rho p k(\beta/\alpha-1)} \|A_2 f\|_{L^p(w)}^p.$$

After taking p-th roots, the resulting double series is summable because $\beta < \alpha$, and this gives strong type.

References

[1] P. Ganguly, T. Rana, and J. Sarkar, *Weighted estimates for Hardy–Littlewood maximal functions on harmonic* NA *groups*, Mathematische Zeitschrift **310**, Article 55 (2025). https://doi.org/10.1007/s00209-025-03762-2.