Subfactors

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July 4, 2026

Abstract

These are notes from the subfactor special lecture given by Emily Peters at the *Groundwork for Operator Algebras Lecture Series* at Michigan State University.

1 Subfactors and the finite group example

Claim 1.1. *A factor has no non-trivial σ-WOT closed two-sided ideals.*

Proof. A σ-WOT closed two-sided ideal in a von Neumann algebra is of the form Mz for a central projection z. Since the center of a factor is $\mathbb{C}1$, this projection is either 0 or 1. Hence the ideal is either 0 or the whole factor. □

We want to understand all maps $f : A \longrightarrow B$, where f is a normal unital $*$-homomorphism and A, B are von Neumann algebras. When A and B are factors,

$$\ker f \text{ is a } \sigma\text{-WOT closed two-sided ideal, so} \qquad \ker f = A \quad \text{or} \quad \ker f = \{0\}.$$

Thus, for $f \neq 0$, $A \cong f(A) \subset B$.

In the unital case, after identifying A with $f(A)$,

$$1_A = 1_B \in A \subset B.$$

Example 1.2. Let M be a II_1 factor with $M \subset B(L^2(M))$ in the standard representation. Suppose $\mathcal{U} < \mathrm{U}(L^2(M))$ is a finite subgroup satisfying $\mathcal{U} \cap M = \{1\}$, $uMu^* = M$, $u\hat{1} = \hat{1}$, for all $u \in \mathcal{U}$.

For $u \in \mathcal{U}$, we write $u(x) = uxu^*$, $x \in M$.

1. Fixed points.
$$M^{\mathcal{U}} = \{x \in M \mid uxu^* = x \ \forall u \in \mathcal{U}\}.$$
If $M^{\mathcal{U}}$ is a factor, then $M^{\mathcal{U}} \subset M$ is a subfactor. Moreover, $\mathrm{tr}_{M^{\mathcal{U}}} = \mathrm{tr}_M \big|_{M^{\mathcal{U}}}$.

2. Crossed product.
$$M \rtimes \mathcal{U} = \left\{ \sum_{u \in \mathcal{U}} x_u u \;\middle|\; x_u \in M \right\}'' \subset B(L^2(M)).$$

The trace is

$$\mathrm{tr}\left(\sum_{u \in \mathcal{U}} x_u u \right) = \mathrm{tr}_M(x_e),$$

where $e \in \mathcal{U}$ is the identity element. The multiplication and involution are given by

$$(x_u u)(y_w w) = x_u \, u(y_w) \, uw,$$

and

$$\left(\sum_{u\in\mathcal{U}} x_u u\right)^* = \sum_{u\in\mathcal{U}} u^* x_u^* = \sum_{u\in\mathcal{U}} u^{-1}(x_u^*)\, u^{-1},$$

where $u^{-1}(x) = u^* x u$. When $M^{\mathcal{U}}$ is a factor, we will show that $M \rtimes \mathcal{U}$ is a factor, and $M \cong M \rtimes \{e\} \subset M \rtimes \mathcal{U}$.

2 Jones index and the basic construction

How do we understand subfactors?

$$1_B \in A \subset B, \qquad A, B \text{ are } \mathrm{II}_1 \text{ factors.}$$

Question. How big is A relative to B?

Let A, B be II_1 factors with $\mathrm{tr}_A = \mathrm{tr}_B\big|_A$. In the standard representation on $L^2(B)$, $L^2(A) \subset L^2(B)$. Let $e_A : L^2(B) \longrightarrow L^2(A)$ be the orthogonal projection.

For the following discussion, A' denotes the commutant of A in $B(L^2(B))$, and $e_A \in A'$.

Definition 2.1.

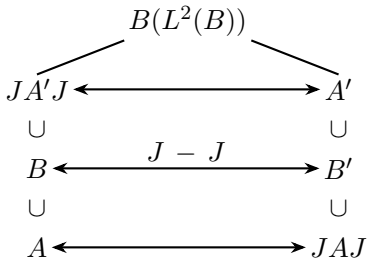
$$[B : A] = \begin{cases} \dfrac{1}{\mathrm{tr}_{A'}(e_A)}, & \text{if } A' \text{ is a } \mathrm{II}_1 \text{ factor,} \\ \infty, & \text{if } A' \text{ is a } \mathrm{II}_\infty \text{ factor.} \end{cases}$$

For the fixed-point subfactor above,

$$[M : M^{\mathcal{U}}] = \frac{1}{\mathrm{tr}_{(M^{\mathcal{U}})'}(e_{M^{\mathcal{U}}})}.$$

Let J be the canonical conjugation on $L^2(B)$.

Idea.



Is there a way to see B, B' as the "same"?

$$B(L^2(B)), \qquad JBJ = B'.$$

What can we say about $JA'J$?

Using $Je_AJ = e_A$,

$$A = \{e_A\}' \cap B = \big(\{e_A\} \cup B'\big)',$$
$$A' = \big(\{e_A\} \cup B'\big)'',$$
$$JA'J = \big(\{e_A\} \cup B\big)''.$$

Definition 2.2 (Jones's basic construction)**.**

$$A \subset B \subset \langle B, e_A\rangle, \qquad \langle B, e_A\rangle := \big(B \cup \{e_A\}\big)''.$$

Equivalently, $\langle B, e_A\rangle = JA'J$.

Theorem 2.3. The algebra $\langle B, e_A\rangle$ is generated by the $*$-algebra $\operatorname{span}\big(B \cup Be_AB\big)$.

Indeed, for $x \in B$, $e_A x e_A = E_A(x)e_A$, where $E_A : B \to A$ is the trace-preserving conditional expectation.

If $[B : A] < \infty$, then $\langle B, e_A\rangle$ is a II_1 factor and

$$[\langle B, e_A\rangle : B] = [B : A].$$

Moreover,

$$\operatorname{tr}_{\langle B,e_A\rangle}(xe_A) = \frac{1}{[B : A]} \operatorname{tr}_B(x) \qquad \forall x \in B.$$

Proposition 2.4. For the finite group example above, if $M^{\mathcal{U}}$ is a factor, then $\langle M, e_{M^{\mathcal{U}}}\rangle = M \rtimes \mathcal{U}$ and $[M : M^{\mathcal{U}}] = |\mathcal{U}|$. Consequently, $M \rtimes \mathcal{U}$ is a II_1 factor.

Proof. Let $p = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} u$. The finite group properties give $p^2 = p$, $p^* = p$.

For $x \in M$,

$$p\hat{x} = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} u\hat{x} = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} uxu^*\hat{1} \in L^2(M^{\mathcal{U}}).$$

If $x \in M^{\mathcal{U}}$, then $p\hat{x} = \hat{x}$. Since $M^{\mathcal{U}}\hat{1}$ is dense in $L^2(M^{\mathcal{U}})$, p is the projection onto $L^2(M^{\mathcal{U}})$. Thus

$$e_{M^{\mathcal{U}}} = p = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} u.$$

Also, $e_{M^{\mathcal{U}}}$ commutes with $M^{\mathcal{U}}$.

By the basic construction theorem, $\langle M, e_{M^{\mathcal{U}}}\rangle$ is generated by $\operatorname{span}(M \cup Me_{M^{\mathcal{U}}}M)$. Since

$$xe_{M^{\mathcal{U}}}y = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} xuy = \frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} x\,u(y)\,u \in M \rtimes \mathcal{U},$$

we get $\langle M, e_{M^{\mathcal{U}}}\rangle \subset M \rtimes \mathcal{U}$. Conversely, let J be the canonical conjugation on $L^2(M)$. If $x \in M$, then $JuJ\hat{x} = Jux^*\hat{1} = Jux^*u^*\hat{1} = uxu^*\hat{1} = u\hat{x}$. Hence $JuJ = u$. Since $\mathcal{U} \subset (M^{\mathcal{U}})'$, we have

$$\mathcal{U} = J\mathcal{U}J \subset J(M^{\mathcal{U}})'J = \langle M, e_{M^{\mathcal{U}}}\rangle.$$

Thus $M \rtimes \mathcal{U} \subset \langle M, e_{M^{\mathcal{U}}}\rangle$, and therefore $\langle M, e_{M^{\mathcal{U}}}\rangle = M \rtimes \mathcal{U}$.

Finally, the trace correspondence between $(M^{\mathcal{U}})'$ and $J(M^{\mathcal{U}})'J = \langle M, e_{M^{\mathcal{U}}}\rangle$, together with $Je_{M^{\mathcal{U}}}J = e_{M^{\mathcal{U}}}$, gives

$$\begin{aligned}
\operatorname{tr}_{(M^{\mathcal{U}})'}(e_{M^{\mathcal{U}}}) &= \operatorname{tr}_{\langle M, e_{M^{\mathcal{U}}}\rangle}(Je_{M^{\mathcal{U}}}J) \\
&= \operatorname{tr}_{\langle M, e_{M^{\mathcal{U}}}\rangle}(e_{M^{\mathcal{U}}}) \\
&= \operatorname{tr}_{M\rtimes\mathcal{U}}\left(\frac{1}{|\mathcal{U}|}\sum_{u\in\mathcal{U}} u\right) \\
&= \frac{1}{|\mathcal{U}|}.
\end{aligned}$$

Therefore, $[M : M^{\mathcal{U}}] = |\mathcal{U}|$.

Since the index is finite, $M \rtimes \mathcal{U} = \langle M, e_{M^{\mathcal{U}}}\rangle$ is a II_1 factor. □

References

[DSN26] R. de Santiago and B. Nelson, *Von Neumann Algebras*, lecture notes for the Groundwork for Operator Algebras Lecture Series, Michigan State University, last updated June 22, 2026.