The Atiyah–Hirzebruch Vanishing Theorem

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Theorem. *Let X be a compact spin manifold admitting a nontrivial S^1-action. Then $\widehat{A}(X) = 0$.*

Proof.

Step 1. If $\dim X$ is odd, then there is nothing to prove: $\widehat{A}(X) = 0$ for degree reasons. So the argument reduces to the even-dimensional case.

Hence we may assume $\dim X = 2m$.

Step 2. Make the action isometric.

Because both X and S^1 are compact, we can average any Riemannian metric over the S^1-action.

After replacing the metric by its average, the S^1-action acts by isometries.

Step 3. Arrange that the action preserves the spin structure.

A spin structure on X is a 2-fold covering $P_{\mathrm{Spin}}(X) \longrightarrow P_{\mathrm{SO}}(X)$.

If a connected Lie group G acts on X, then either G itself or a 2-fold covering group of G preserves the spin structure.

Since a 2-fold covering of S^1 is again S^1, we may replace the given action by that lifted circle action.

Assume from now on that the S^1-action preserves the spin structure.

Step 4. The action on spinors, equivariant index.

Because the action is by isometries and preserves the spin structure, the S^1-action lifts to an action on the complex spinor bundle $\mathcal{S}_{\mathbb{C}} = \mathcal{S}^+_{\mathbb{C}} \oplus \mathcal{S}^-_{\mathbb{C}}$. Since S^1 is connected, it preserves the orientation and hence the chiral splitting. This lifted action commutes with the AS operator

$$\not{D}^+ : \Gamma(\mathcal{S}^+_{\mathbb{C}}) \longrightarrow \Gamma(\mathcal{S}^-_{\mathbb{C}}).$$

Hence the finite-dimensional spaces $\ker \not{D}^+$ and $\ker \not{D}^-$ are complex S^1-representations.

Then each of them decomposes into weight spaces

$$\ker \not{D}^{\pm} = \bigoplus_{k \in \mathbb{Z}} (\ker \not{D}^{\pm})_k,$$

where

$$(\ker \not{D}^{\pm})_k = \{v \in \ker \not{D}^{\pm} \mid z \cdot v = z^k v \quad \forall z \in S^1\}.$$

Equivalently, every irreducible complex representation of S^1 is one-dimensional: $\forall n \in \mathbb{Z}$, $\exists$ a representation $\rho_n : S^1 \longrightarrow \mathrm{GL}(1,\mathbb{C})$, $e^{i\theta} \longmapsto e^{in\theta}$.

So any finite-dimensional S^1-representation V decomposes as $V \cong \bigoplus_k \rho_{n_k}$.

In particular, $\ker \not{D}^+ = \bigoplus_k V_k^+$, where $V_k^+ = \{v \in \ker \not{D}^+ : z \cdot v = z^k v \quad \forall z \in S^1\}$.

Similarly, $\ker \not{D}^- = \bigoplus_k V_k^-$, where $V_k^- = \{v \in \ker \not{D}^- : z \cdot v = z^k v \quad \forall z \in S^1\}$.

Now define the equivariant index. Since $\not{D}^+$ is S^1-equivariant, $\ker \not{D}^+$ and $\ker \not{D}^-$ are both finite-dimensional S^1-representations, we may form the equivariant index:

$$\mathrm{Ind}_{S^1}[\not{D}^+] = [\ker \not{D}^+] - [\ker \not{D}^-] \in R(S^1).$$

Under the character isomorphism $R(S^1) \cong \mathbb{Z}[t,t^{-1}]$, with $[\rho_n] \longmapsto t^n$, $t = e^{i\theta}$, and representation $\longmapsto$ trace, the equivariant index becomes the Laurent polynomial $i_{\not{D}^+}(t) = \sum_k (\dim V_k^+ - \dim V_k^-)t^k$.

Note $\forall g \in S^1$, $\mathrm{tr}(g \mid \ker \not{D}^+) = \sum_k (\dim V_k^+) g^k$, because on a space where g acts by the scalar g^k, the trace is just $g^k \cdot$ dimension.

Similarly, $\mathrm{tr}(g \mid \ker \not{D}^-) = \sum_k (\dim V_k^-) g^k$. Then

$$\mathrm{tr}(g \mid \ker \not{D}^+) - \mathrm{tr}(g \mid \ker \not{D}^-) = \sum_k (\dim V_k^+ - \dim V_k^-) g^k = i_{\not{D}^+}(g).$$

If we are evaluating at $g = 1$, we recover $i_{\not{D}^+}(1) = \dim \ker \not{D}^+ - \dim \ker \not{D}^- = \widehat{A}(X)$.

So if we can show $i_{\not{D}^+} = 0$ as a Laurent polynomial, then $\widehat{A}(X) = 0$.

Step 5. Choose a non-torsion element and identify the fixed point set.

Choose $g \in S^1$ which is not a root of unity. Then the subgroup generated by g is dense in S^1: $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$.

Let

$$\bar{F}_g = X^g = \{x \in X : g \cdot x = x\}, \qquad \bar{F} = X^{S^1} = \{x \in X : z \cdot x = x \quad \forall z \in S^1\}.$$

We claim that $\bar{F}_g = \bar{F}$.

$\bar{F} \subseteq \bar{F}_g$: if $x \in \bar{F} = X^{S^1}$, then $z \cdot x = x \ \forall z \in S^1$. Since $g \in S^1$, $g \cdot x = x$, so $x \in \bar{F}_g$.

$\bar{F}_g \subseteq \bar{F}$: if $x \in \bar{F}_g$, then $g^n \cdot x = x \ \forall n \in \mathbb{Z}$.

Since $\langle g \rangle$ is dense in S^1, for any $h \in S^1$, $\exists$ a sequence $n_1, \cdots, n_j$ such that $g^{n_j} \longrightarrow h$ in S^1. By continuity of the action,

$$h \cdot x = \lim_{j \to \infty} g^{n_j} \cdot x = \lim_{j \to \infty} x = x.$$

h was chosen arbitrarily, so $x \in \bar{F}$, and therefore $\bar{F}_g = \bar{F}$.

Step 6. G-spin theorem.

Let N be the normal bundle of $\bar{F}$ in X. Because $\bar{F}$ is fixed by S^1, the S^1-action preserves N. N decomposes S^1-equivariantly as $N = \bigoplus_{m>0} N_m$, where each N_m is a complex vector bundle. When $g \in S^1$ acts on $v \in N_m$, $g \cdot v = g^m \cdot v$, where $g^m \in \mathbb{C}$ ($g \in S^1 \subset \mathbb{C}$).

We want to compute $i_{\not D^+}(g)$ for non-torsion g.

For a complex rank k bundle E with formal Chern roots $x_1, \cdots, x_k$, define the characteristic function

$$V(E,t) = \prod_{j=1}^{k} \left(t^{1/2} e^{x_j/2} - t^{-1/2} e^{-x_j/2} \right)^{-1}.$$

Because

$$\widehat{A}_\theta(E) = \prod_{j=1}^{k} \left(\underbrace{e^{\frac{1}{2}(x_j+i\theta)}}_{:=①} - \underbrace{e^{-\frac{1}{2}(x_j+i\theta)}}_{:=②} \right)^{-1},$$

if we set $t = e^{i\theta}$, then $V(E,t)$ is the characteristic class $\widehat{A}_\theta(E)$ rewritten in $t = e^{i\theta}$.

Because $\bar{F}_g = \bar{F}$, the formula becomes, componentwise on $\bar{F}$,

$$\mathrm{Spin}(X,g) = (-1)^\delta \left\{ \prod_{0<\theta\leq\pi} \widehat{A}_\theta(N(\theta)) \cdot \widehat{A}(\bar{F}) \right\} [\bar{F}],$$

where $(-1)^\delta$ is locally constant on the components of $\bar{F}$.

In the G-spin theorem, the normal bundle decomposes by rotation angle: $N = \bigoplus N(\theta)$, and dg rotates vectors in $N(\theta)$ by angle θ.

In our case, $g = e^{i\theta_0} \in S^1$ acts on N_m by $g^m = e^{im\theta_0}$. So the factor coming from N_m is rewritten by setting $e^{i\theta} = g^m$.

From $\widehat{A}_\theta(E) = V(E, e^{i\theta})$, we compute the corresponding factor as

$$V(N_m, e^{im\theta_0}) = V(N_m, g^m).$$

By definition,

$$\mathrm{Spin}(X,g) = \mathrm{ind}_g(\not D^+) = i_{\not D^+}(g) = (-1)^\delta \left\{ \prod_m V(N_m, g^m) \widehat{A}(\bar{F}) \right\} [\bar{F}]. \qquad (*)$$

LHS: $i_{\not D^+}(g)$ is the Laurent polynomial $\sum_k a_k t^k$ evaluated at g, in particular a rational function.

RHS: Look at $V(N_m, g^m)$. Each factor is $\left(g^{m/2} e^{x_j/2} - g^{-m/2} e^{-x_j/2}\right)^{-1}$. Although $V(N_m, g^m)$ is defined only up to sign because of the square roots, the G-spin theorem fixes the signs on the components of $\bar{F}$. After multiplying by $\widehat{A}(\bar{F})$ and evaluating on $[\bar{F}]$, the whole RHS gives the rational function in g which agrees with $i_{\not D^+}(g)$.

Both sides of $(*)$ are rational functions of g. This identity holds for all non-torsion g. We replace g by a variable t.

Step 7. WTS: RHS vanishes as $t \to 0$ and as $t \to \infty$.

Case ①: If $\bar{F} = \varnothing$, RHS $= 0$ for all non-torsion g, so the Laurent polynomial $i_{\not D^+}(t)$ is identically zero, and we are done.

Case ②: Suppose $\bar{F} \neq \varnothing$. Since the S^1-action is nontrivial, $\bar{F} \neq X$; otherwise every point would be fixed and the action would be trivial.

So $\bar{F}$ is a proper submanifold of X, $\dim \bar{F} < \dim X$, and the normal bundle N is nonzero. Since $N = \bigoplus_{m>0} N_m$ as a real bundle with complex summands and $\operatorname{rank}_{\mathbb{R}}(N) = 2\sum_{m>0} \operatorname{rank}_{\mathbb{C}}(N_m) > 0$, at least one N_m has positive complex rank.

Now we are looking at $V(E,t)$.

$$\begin{aligned} V(E,t) &= \prod_{j=1}^{k} \left(t^{1/2} e^{x_j/2} - t^{-1/2} e^{-x_j/2} \right)^{-1} \\ &= \prod_{j=1}^{k} t^{1/2} e^{x_j/2} \left(t e^{x_j} - 1 \right)^{-1} \qquad ① \\ &= \prod_{j=1}^{k} t^{-1/2} e^{-x_j/2} \left(1 - t^{-1} e^{-x_j} \right)^{-1} \qquad ②. \end{aligned}$$

$$\left.\begin{array}{ll} ①: \text{ if } t \to 0, & t^{1/2} e^{x_j/2} \longrightarrow 0, \\ ②: \text{ if } t \to \infty, & t^{-1/2} e^{-x_j/2} \longrightarrow 0, \end{array}\right\} \quad \text{whenever } \operatorname{rank}_{\mathbb{C}}(E) > 0.$$

Thus $V(E,t) \longrightarrow 0$ as $t \to 0$ and as $t \to \infty$. We apply this to at least one positive complex-rank summand N_m, so $\prod_m V(N_m, t^m)$ vanishes as $t \to 0$ and as $t \to \infty$.

So the RHS vanishes at both limits: $i_{\not D^+}(t) \longrightarrow 0$ as $t \to 0$ and as $t \to \infty$.

If $i_{\not D^+}(t)$ is not identically zero, write $i_{\not D^+}(t) = \sum_{j=p}^{q} a_j t^j$ with $a_p \neq 0$ and $a_q \neq 0$.

$$\begin{array}{ll} \text{If } p < 0, & a_p t^p \to \infty \text{ as } \boxed{t \to 0}, \text{ but } i_{\not D^+}(t) \to 0, \\ p = 0, & i_{\not D^+}(t) \to a_0 \neq 0, \end{array} \qquad \Longrightarrow \qquad p > 0.$$

$$\begin{array}{ll} \text{If } q > 0, & a_q t^q \to \infty \text{ as } \boxed{t \to \infty}, \text{ but } i_{\not D^+}(t) \to 0, \\ q = 0, & i_{\not D^+}(t) \to a_0 \neq 0, \end{array} \qquad \Longrightarrow \qquad q < 0.$$

But $p \leq q$ always. Therefore $i_{\not D^+}(t) \equiv 0$.

Finally, $0 = i_{\not D^+}(1) = \dim\ker \not D^+ - \dim\ker \not D^- = \widehat{A}(X)$.

□

References

[1] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn, *Spin Geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.