Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number.

Define the rotation $R_\theta(z) := e^{2\pi i\theta} z$. Then $\mathbb{Z}$ acts on S^1 by iterating R_θ:

$$n \cdot z := R_\theta^n(z) = e^{2\pi i n\theta} z.$$

Our goals are to understand:

1. the orbit quotient $S^1/\mathbb{Z}$,
2. the crossed product $C(S^1) \rtimes_\alpha \mathbb{Z}$,
3. the transformation groupoid $S^1 \rtimes_\theta \mathbb{Z}$.

1. Let $z \in S^1$. Consider the subgroup $H_\theta = \{e^{2\pi i n\theta} : n \in \mathbb{Z}\} \subset S^1$. Then the orbit of z is

$$O(z) = zH_\theta = \{ze^{2\pi i n\theta} : n \in \mathbb{Z}\},$$

so it is enough to understand the closure of H_θ.

Let $\overline{H_\theta}$ be the closure of H_θ in S^1. Since the closure of a subgroup in a topological group is again a subgroup, $\overline{H_\theta}$ is a closed subgroup of S^1.

Recall: a closed subgroup of S^1 is either finite cyclic or S^1.

Since θ is irrational, $e^{2\pi i n\theta} \neq 1$ for all $n \neq 0$. So H_θ is not finite. Then $\overline{H_\theta}$ is not finite, so $\overline{H_\theta} = S^1$.

Thus H_θ is dense, and therefore every orbit zH_θ is dense in S^1.

Intuitively, this already tells us that the quotient space is BAD. But why?

Let $q : S^1 \to S^1/\mathbb{Z}$ be the orbit map. Take a nonempty closed subset $F \subset S^1/\mathbb{Z}$. Then $q^{-1}(F) \subset S^1$ is closed by the definition of the quotient topology.

If $x \in q^{-1}(F)$ and y lies in the same orbit as x, then $q(y) = q(x) \in F$, so $y \in q^{-1}(F)$. This is to say that $q^{-1}(F)$ is saturated.

Now choose $z \in q^{-1}(F)$. Since we just showed $q^{-1}(F)$ is saturated, it contains $O(z)$. But $O(z)$ is dense in S^1, and $q^{-1}(F)$ is closed. Thus, $q^{-1}(F) = S^1$. Hence $F = S^1/\mathbb{Z}$. So the only nonempty closed subset of $S^1/\mathbb{Z}$ is the whole space. So the quotient topology is TRIVIAL! :((that is indiscrete) [1, Sec. 3.5]

Topologically it behaves like a collapsed object. This answers our earlier intuition.

2. Now we pass to the C^*-algebra. Since S^1 is compact, $C_0(S^1) = C(S^1)$. The action on the space induces an action on functions $\alpha_n(f) = f \circ R_\theta^{-n}$. Explicitly, if we want to look at action on points, $\alpha_n(f)(z) = f(e^{-2\pi i n\theta} z)$.

Remark: here we are also discussing a dynamical system in the sense of [1, Sec. 2.1].

The inverse is there because functions transform contravariantly.

Now we check the group law,

$$\begin{aligned}\alpha_m(\alpha_n(f)) &= \alpha_m(f \circ R_\theta^{-n}) \\ &= (f \circ R_\theta^{-n}) \circ R_\theta^{-m} \\ &= f \circ R_\theta^{-(m+n)} \\ &= \alpha_{m+n}(f),\end{aligned}$$

so $\alpha : \mathbb{Z} \to \mathrm{Aut}(C(S^1))$ is a homomorphism.

Now we introduce a unitary u implementing the action, so that

$$ufu^* = \alpha_1(f), \qquad f \in C(S^1);$$

this is to say $u^m f u^{-m} = \alpha_m(f)$, $m \in \mathbb{Z}$.

The algebraic crossed product is the *-algebra of finite sums $\sum_{n\in\mathbb{Z}} f_n u^n$, $f_n \in C(S^1)$ with only finitely many $f_n \neq 0$.

We check product rule: Take monomials fu^m and gu^n, then we compute

$$\begin{aligned}(fu^m)(gu^n) &= f(u^m g u^{-m})u^{m+n} \\ &= f\,\alpha_m(g)\,u^{m+n} \qquad \text{(Using the covariance relation).}\end{aligned}$$

For finite sums, $a = \sum_m f_m u^m$, $b = \sum_n g_n u^n$, we get

$$ab = \sum_{m,n} f_m \alpha_m(g_n) u^{m+n} = \sum_k \left(\sum_m f_m \alpha_m(g_{k-m}) \right) u^k.$$

Involution: we start with taking a monomial fu^n. Then,

$$\begin{aligned}(fu^n)^* &= (u^n)^* f^* \\ &= u^{-n}\bar{f} \\ &= (u^{-n}\bar{f}u^n)u^{-n} \qquad \text{since } u^n u^{-n} = 1 \\ &= \alpha_{-n}(\bar{f})u^{-n}.\end{aligned}$$

Again if we take $a = \sum_n f_n u^n$, we get

$$a^* = \sum_n \alpha_{-n}(\overline{f_n})u^{-n} = \sum_k \alpha_k(\overline{f_{-k}})u^k.$$

Clarification: $C(S^1) \rtimes_\alpha \mathbb{Z} \neq C(S^1)^{\mathbb{Z}}$. Because every orbit is dense, any continuous invariant function is constant. So $C(S^1)^{\mathbb{Z}} = \mathbb{C}$.

Now we should recover the irrational rotation algebra relation. We take the coordinate function $v \in C(S^1)$, $v(z) = z$. Because $|z| = 1$ on S^1, v is a unitary in $C(S^1)$.

We compute that

$$\begin{aligned}\alpha_1(v)(z) &= v(e^{-2\pi i\theta}z)\\ &= e^{-2\pi i\theta}z\\ &= e^{-2\pi i\theta}v(z),\end{aligned}$$

which is to say

$$uvu^* = e^{-2\pi i\theta}v.$$

Now let $U = u^*$ and $V = v$, so $U^* = u$. Then

$$U^*V = e^{-2\pi i\theta}VU^* \implies V = e^{-2\pi i\theta}UVU^* \implies UV = e^{2\pi i\theta}VU.$$

Equivalently,

$$UV = e^{2\pi i\theta}VU,$$

so $A_\theta \cong C(S^1) \rtimes_\alpha \mathbb{Z}$ is generated by unitaries satisfying this relation [1, Example 2.55 and Proposition 2.56].

3. Lastly we want to discuss the groupoid formulation.

Define the transformation groupoid $\mathcal{G} = S^1 \rtimes_\theta \mathbb{Z}$. As a set, $\mathcal{G} = S^1 \times \mathbb{Z}$.

(z, n) is the arrow $R_\theta^{-n}(z) \to z$. So the structure maps are

$$\mathrm{t}(z, n) = z, \quad \mathrm{s}(z, n) = R_\theta^{-n}(z) = e^{-2\pi in\theta}z,$$

the multiplication is determined by composition of arrows,

$$(z, n) \cdot (R_\theta^{-n}(z), m) = (z, n + m),$$

and the inverse is

$$(z, n)^{-1} = (R_\theta^{-n}(z), -n).$$

Since $\mathbb{Z}$ is discrete, each slice $S^1 \times \{n\}$ is open in $\mathcal{G}$. On this slice, $\mathrm{t}(z, n) = z$ is a homeomorphism onto S^1, and $\mathrm{s}(z, n) = R_\theta^{-n}(z)$ is also a homeomorphism onto S^1. Both t and s are local homeomorphisms, so $\mathcal{G}$ is étale.

The isotropy at z consists of arrows (z, n) with $\mathrm{t}(z, n) = \mathrm{s}(z, n) = z$. So,

$$e^{-2\pi in\theta}z = z \iff e^{-2\pi in\theta} = 1 \iff n\theta \in \mathbb{Z}.$$

Since θ is irrational, this implies $n = 0$. Therefore isotropy is trivial, so the groupoid is principal. Because every orbit in S^1 is dense, the groupoid is also minimal.

The next step we want to do is to compute $C_c(\mathcal{G})$ and we want to identify it with the crossed product.

A compactly supported continuous function on $\mathcal{G} = S^1 \times \mathbb{Z}$ is a function $F : S^1 \times \mathbb{Z} \to \mathbb{C}$ such that only finitely many integers n occur in its support. So we may write $F_n(z) := F(z, n)$, $F_n \in C(S^1)$, and only finitely many F_n are nonzero. Thus,

$$C_c(\mathcal{G}) \cong C_c(\mathbb{Z}, C(S^1)).$$

Convolution: for an étale groupoid, convolution is $(F * H)(\gamma) = \sum_{\mathrm{t}(\eta)=\mathrm{t}(\gamma)} F(\eta)H(\eta^{-1}\gamma)$ [2, Sec. 2.2]. Take $\gamma = (z, n)$. Then $\mathrm{t}(\gamma) = z$. The arrows η with $\mathrm{t}(\eta) = z$ are $\eta = (z, m)$, $m \in \mathbb{Z}$. So,

$$\begin{aligned}(F * H)(z, n) &= \sum_{m\in\mathbb{Z}} F(z, m)H((z, m)^{-1}(z, n)) \\ &= \sum_{m\in\mathbb{Z}} F(z, m)H(R_\theta^{-m}(z), n - m) \\ &= \sum_{m\in\mathbb{Z}} F_m(z)H_{n-m}(R_\theta^{-m}(z)) \\ &= \sum_{m\in\mathbb{Z}} F_m(z)\alpha_m(H_{n-m})(z), \qquad \text{as } \alpha_m(g)(z) = g(R_\theta^{-m}(z)).\end{aligned}$$

Hence, $(F * H)_n = \sum_{m\in\mathbb{Z}} F_m\alpha_m(H_{n-m})$.

Involution: For groupoid C_c-algebras, $F^*(\gamma) = \overline{F(\gamma^{-1})}$. So $F^*(z, n) = \overline{F((z, n)^{-1})} = \overline{F(R_\theta^{-n}(z), -n)}$. Thus,

$$(F^*)_n(z) = \overline{F_{-n}(R_\theta^{-n}(z))} = \alpha_n(\overline{F_{-n}})(z),$$

Note that this matches the coefficient form of the algebraic crossed product involution $(\sum_n F_n u^n)^* = \sum_n \alpha_n(\overline{F_{-n}})u^n$. So the identification $F \leftrightarrow (F_n)_{n\in\mathbb{Z}}$ is a *-algebra isomorphism $C_c(S^1 \rtimes_\theta \mathbb{Z}) \cong C_c(\mathbb{Z}, C(S^1))$, and after completion,

$$C^*(S^1 \rtimes_\theta \mathbb{Z}) \cong C(S^1) \rtimes_\alpha \mathbb{Z}.$$

References

[1] Dana P. Williams. *Crossed Products of C*-Algebras*. Preliminary version, January 31, 2006. 2006.

[2] Yuezhao Li. *Groupoid C*-algebras*. Leiden Noncommutative Geometry Seminar notes. 2024.